Theta Ciphering Nationals 2024 solutions

0. $\dfrac{-4}{n} = \dfrac{3}{n-1} \to -4n+4 = 3n \to 7n = 4 \to n = \dfrac{4}{7}$ 7n=4

1. $f(x) = \sqrt{-3x^2 + 24x + 1} = \sqrt{-3(x-4)^2 + 49}$ so it is easy to see we have a parabola opening

   down. The range is [0,7]. The domain is $\left[4 - \dfrac{7\sqrt{3}}{3}, 4 + \dfrac{7\sqrt{3}}{3} \right]$ so 4+4+0+7=15

2. Imaginary roots come in conjugate pairs so 1-5i is a root. The sum of roots is 6 so the third root
   is 4.
   $f(x) = x^3 - 6x^2 + Rx + S = 0 = (x-1-5i)(x-1+5i)(x-4)$
   $(x^2 - 2x + 26)(x-4) \to R = 34 \to S = -104 \to 34 - 104 = -70$

3. $x^2 + y^2 = r^2 \to (r-4)^2 + 100 = r^2 \to -8r + 16 + 100 = 0$
   $8r = 116 \to r = 14.5 = \dfrac{29}{2}$

4. $k^4 - 4k^3 + 6k^2 - 4k + 1 = k^4 - 4k^3 + k + 5$
   $6k^2 - 5x - 4 = 0 = (3k-4)(2k+1) \to k = \dfrac{4}{3}$ k=-1/2 is extraneous

5. $\sqrt{\log k} = \log \sqrt{k} \to \log k = \dfrac{1}{4}(\log k)^2 \to \dfrac{1}{4}(\log k)^2 - \log k = 0$
   $\log k (\log k - 4) = 0 \to k = 1, 10,000 \to 10,001$

6. This is a cylinder with a radius of 4 and a height of 2. $2\pi rh + 2\pi r^2 = 16\pi + 32\pi = 48\pi$

7. $-15 \le k \le 31$ which is summing integers from 16 to 31 inclusive $\frac{16}{2}(16+31) = 8 \bullet 47 = 376$

8. If you synthetically divide you get $k+5+\dfrac{20}{k-3}$ so 23 is the largest that makes an integer

9. If you draw a segment bisecting the 30-degree angle you create a parallelogram. If we call WZ 5x then X to this point is 5x and that point to Y is 4x. Then you can use angle bisector theorem to get 12 for our intended segment

10. $90+60+\dfrac{3}{4}(30) = \dfrac{345}{2} \to 8\pi\dfrac{\frac{345}{2}}{360} = \dfrac{23\pi}{6}$

11. $y = x^2 - 10x + 25 + 12 = (x-5)^2 + 12 \to (5,12)$ the distance to the origin is 13 and the distance from (14,0) is 15. Major axis has length of 2a which is 13+15=28

12. $2024 = 2^3 \bullet 11 \bullet 23 \to 4 \bullet 2 \bullet 2 = 16$. We want the average of the 8th and 9th factor when listed in order 1,2,4,8,11,22,23,44,46,88,92,…2024. average of 44,46 = 45

Answers:

0. 4
1. 15
2. -70
3. $14.5 = \dfrac{29}{2}$
4. $\dfrac{4}{3}$
5. 10,001
6. 48π
7. 376
8. 23
9. 12
10. $\dfrac{23\pi}{6}$
11. 28
12. 45