Question 0

A: A diameter of $2\sqrt{2}$ means a radius of $\sqrt{2}$. $A = \pi r^2 = \pi\left(\sqrt{2}\right)^2 = \boxed{2\pi}$.

B: The period of a sinusoid is $\frac{2\pi}{B}$, where $f(x) = A\sin(Bx) + C$. The period is $\frac{2\pi}{2\pi} = \boxed{1}$.

C: $f'(x) = 4x^3 + 8x$. $f'(1) = 4 + 8 = \boxed{12}$.

Question 1

A: The constant term is $\binom{6}{2}(x^2)^2\left(-\frac{1}{x}\right)^4 = \boxed{15}$.

B: The number of terms in $(x + y + 1)^A$ is equivalent to the number of nonnegative integer solutions to $a + b + c = A$. By stars and bars, this is equal to $\binom{A+2}{2}$, which at $A = 15$ is 136. The sum of digits is $\boxed{10}$.

C: The $x^{-\frac{3}{2}}$ term is $\binom{\frac{1}{2}}{2}x^{-\frac{3}{2}}B^2 = -\frac{B^2}{8}(x)^{-\frac{3}{2}}$. At $B = 10$, the coefficient is equal to $-\frac{100}{8} = \boxed{-\frac{25}{2}}$.

Question 2

A: Critical values are at $x = 3, x = \frac{4}{3}$.

At $x \geq 3$, $2x - 6 + 3x - 4 = 8, 5x - 10 = 8, 5x = 18, x = \frac{18}{5}$ which is valid.
At $\frac{4}{3} \leq x \leq 3$, $6 - 2x + 3x - 4 = 8$, $x = 6$, which is extraneous.
At $x \leq \frac{4}{3}$, $6 - 2x - 3x + 4 = 8$, $2 = 5x$, $x = \frac{2}{5}$ which is also valid. The answer is therefore $\boxed{4}$.

B: $\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \frac{A + \frac{11}{60}}{1 - \frac{11A}{60}} = \frac{60A + 11}{60 - 11A}$. At $A = 4$, $\tan(\alpha - \beta) = \frac{251}{16}$, meaning $B = \boxed{267}$.

C: There are 4 cases to consider for the derivatives, because they cycle every 4.

1. $B \equiv 0 (mod\ 4)$

$$f^B(x) = 3^B e^{3x} - 4^B \sin(4x) + 2^B \cos(2x)$$

At 0, this evaluates to $3^B + 2^B$. The last digit of 3^B is always 1, and the last digit of 2^B is always 6 when $B \equiv 0(mod\ 4), B > 0$, so the answer would be 7 if $B \equiv 0(mod\ 4)$.

2. $B \equiv 1(mod\ 4)$

$$f^B(x) = 3^B e^{3x} - 4^B \cos(4x) - 2^B \sin(2x),$$

Which at 0 evaluates to $3^B - 4^B$. However, this is always negative for $B \geq 1$, which means that $|f^B(0)| = 4^B - 3^B$. In this case, the last digit of 4^B and 3^B are 4 and 3, respectively, which means that the answer will be 1 if $B \equiv 1(mod\ 4)$.

3. $B \equiv 2(mod\ 4)$

$$f^B(x) = 3^B e^{3x} + 4^B \sin(4x) - 2^B \cos(2x)$$

Which evaluates to $3^B - 2^B$ at $x = 0$. The last digit of 3^B and 2^B here are 9 and 4, respectively, which means that the answer would be 5 if $B \equiv 2(mod\ 4)$.

4. $B \equiv 3\ (mod\ 4)$

$$f^B(x) = 3^B e^{3x} + 4^B \cos(4x) + 2^B \sin(2x),$$

Which at 0 evaluates to $3^B + 4^B$. The last digit of 3^B and 4^B are 7 and 4, respectively, meaning that the answer would be 1 if $B \equiv 3(mod\ 4)$.

Because $B = 267 \equiv 3(mod\ 4)$, the answer is $\boxed{1}$.

Question 3

A: Apply Heron's formula, $A = \sqrt{35(35-28)(35-17)(35-25)} = \sqrt{35 \cdot 7 \cdot 18 \cdot 10} = \boxed{210}$

B: Let θ be the enclosed angle between the sides of lengths 25 and 28. We have $\frac{1}{2} \cdot 25 \cdot 28 \cdot \sin\theta = A$, so $\sin\theta = \frac{A}{350}$. Applying law of cosines, we have $x^2 = 25^2 + 28^2 - 2 \cdot 25 \cdot 28 \cdot \cos\theta = 1409 - 1400\cos\theta$. Since we want the larger value of x^2, we want to use the negative value of $\cos\theta$.

With $A = 210, \sin\theta = \frac{3}{5}, \cos\theta = -\frac{4}{5}$. Substituting in, we have $B = 1409 + 1400 \cdot \frac{4}{5} = \boxed{2529}$.

C: Let x be the area of the triangle, and a, b, c be its sides, and s be its semiperimeter. By Heron's formula, $x^2 = s(s-a)(s-b)(s-c)$, differentiating implicitly, and plugging in $x = 6, s = 6, s - a = 3, s - b = 2, s - c = 1$, we have

$$2(6) \cdot \frac{dx}{dt} = \frac{ds}{dt}(6) + \frac{d(s-a)}{dt}(12) + \frac{d(s-b)}{dt}(18) + \frac{d(s-c)}{dt}(36)$$

Note that $\frac{da}{dt} = \frac{db}{dt} = \frac{dc}{dt} = 2C$, so $\frac{ds}{dt} = \frac{1}{2}(3 \cdot 2C) = 3C$, and $\frac{d(s-a)}{dt} = \frac{d(s-b)}{dt} = \frac{d(s-c)}{dt} = C$.

We have $12B = 18C + 12C + 18C + 36C$, or $C = \frac{B}{7} = \boxed{\frac{2529}{7}}$.

Question 4

A: $a(t) = 2t - 8, v(t) = \int a(t)dt = t^2 - 8t + C$. Since $v(0) = -9, C = -9$. $t^2 - 8t - 9$ can be factored as $(t-9)(t+1)$. We want the intervals where the velocity and acceleration are the same sign. From $-1 \le t \le 9$, $v(t)$ is negative, and everywhere else, $v(t)$ is positive. For acceleration, anything below $t = 4$ is negative, and anything above is positive. This means that the intervals $(0,4)$ and $(9,12)$ are when the speed of the particle increases, which has a length of $\boxed{7}$ units.

B: Assuming the two points are on the same side of the line, we want to reflect one of the two points over the line and calculate the straight-line distance between the other point and this reflection (If the points were on different sides of the line, calculating the straight-line distance is enough). Reflecting over $y = x$ changes the x and y coordinate values, meaning that if Alan was at $(15, -1)$, he will now be at $(-1, 15)$. The distance between Sam and Alan is then $\sqrt{(A+1)^2 + 15^2}$, which evaluated at $A = 7$ is $\boxed{17}$.

C: First, we can solve for the 2nd row, 3rd column of M^{-1} in terms of x and B. Since $2 + 3 = 5$, which is odd, we negate the minor of the 3rd row, 2nd column and divide by the determinant of the matrix. This means that our answer, in terms of x and B, is

$$-\frac{-4 \cdot x - 0 \cdot 2}{B} = \frac{4x}{B}.$$

To solve for x in terms of B, we find the determinant of the matrix in terms of x and set it equal to B. The value of the determinant is $8x^2 + 18x + 12$, so

$$8x^2 + 18x + 12 = B.$$

At $B = 17$, this means

$$8x^2 + 18x - 5 = 0,$$
$$(4x - 1)(2x + 5) = 0,$$
$$x = \frac{1}{4}, -\frac{5}{2}.$$

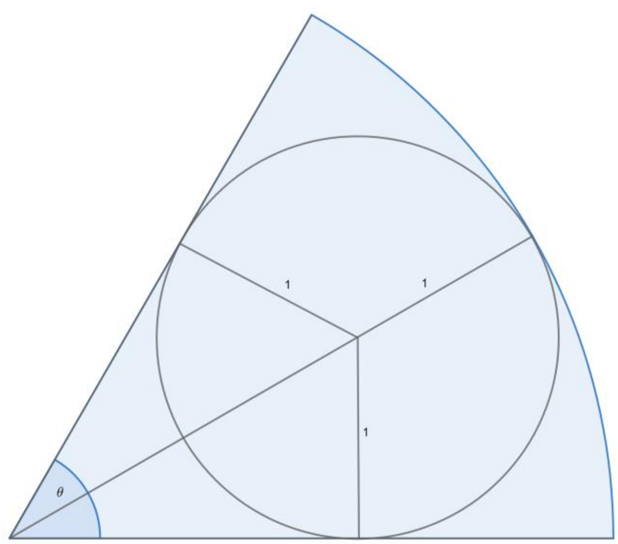
Taking the positive value of x, we get

$$C = \boxed{\frac{1}{4}}.$$

Question 5

A: Using L'Hospital, $\lim_{x\to 0}\left(\frac{\sin 2x}{x} + \frac{\arcsin 2x}{x}\right) = \lim_{x\to 0}\left(\frac{2\cos 2x}{1} + \frac{\frac{2}{\sqrt{1-4x^2}}}{1}\right) = \boxed{4}.$

B:



Draw a good picture. Let the angle of the sector be θ. This means that, when the line connecting the center of the sector with the center of the circle is drawn, that it splits θ in half. Let the segment connecting the two centers be x. This means that

$$x \sin\frac{\theta}{2} = A,$$
$$x = \frac{A}{\sin\frac{\theta}{2}}.$$
The radius of the sector will then be $A + x = A + \frac{A}{\sin\frac{\theta}{2}}$.

At $\theta = 60°$, $\sin\frac{\theta}{2} = \sin 30° = \frac{1}{2}$, so the radius of the sector when $A = 4$ is $4 + \frac{4}{\frac{1}{2}} = 4 + 8 =$ $\boxed{12}$.

C: Put this on the argand plane. Our sum becomes an infinite geometric series. For the sake of convenience, we will lump two steps together so the common ratio is real. The first term is $B + \frac{Bi}{2}$ and common ratio is $-\frac{1}{4}$. Using the infinite geometric series formula, we can find the complex number value of the point at which Potato ends up.

$$x + yi = \frac{B + \frac{Bi}{2}}{1 - \left(-\frac{1}{4}\right)}.$$

At $B = 12$ This value is $\frac{12+6i}{1+\frac{1}{4}} = \frac{12+6i}{\frac{5}{4}} = \frac{48+24i}{5}$.

Since $x = \frac{48}{5}$ and $y = \frac{24}{5}$, $x + y = \boxed{\frac{72}{5}}$.

Question 6

A: Since $9 = 3^2, 9^{15} = 3^{30}$. We must find the number of ways to express 3^{30} as a^b, with a and b being integers. b must be a positive integer factor of 30, so now we can do casework on b.

Case 1: b is even

When b is even, a can be positive or negative, which means that there are exactly 2 values of a for every even factor of 30 b can be. Since 30 has 4 even factors, this case yields $4 \cdot 2 = 8$ solutions.

Case 2: b is odd

When b is odd, a can only be positive, which means that there is exactly 1 value of a for every odd b. Since 30 has 4 odd factors, this case yields 4 solutions.

Our final answer is $8 + 4 = \boxed{12}$.

B: At $x = 1$, there are $A - 2$ lattice points inside this triangle. At $x = 2$, there are $A - 3$. This pattern continues all the way down to 1 lattice point at $x = A - 1$. This means that the answer is the sum of the first $A - 2$ positive integers, which evaluates to $\frac{(A-2)(A-1)}{2}$. At $A = 12$, This equals $\boxed{55}$.

C: Rewrite the line in parametric form, we have $x = t + 1, y = t + 2, z = t - 6$. Then the square of the distance from the point $(5, 5, 5)$ to an arbitrary point on the line is

$$(t + 1 - 5)^2 + (t + 2 - 5)^2 + (t - 6 - 5)^2 = 3t^2 - 36t + 146$$

This quantity is minimized when $t = 6$, so the shortest distance is $\sqrt{3 \cdot 6^2 - 36 \cdot 6 + 146} = \boxed{\sqrt{38}}$

Question 7

A: $A = \text{sum of areas of two smaller semicircles} + [XYZ] - \text{area of semicircle} = \frac{1}{2}\left(\left(\frac{XZ}{2}\right)^2 \pi + \left(\frac{YZ}{2}\right)^2 \pi + 10 \cdot 4 - \left(\frac{XY}{2}\right)^2 \pi\right) = \frac{XZ^2 + YZ^2 - XY^2}{8}\pi + 20$. However, by the Pythagorean theorem, $XZ^2 + YZ^2 - XY^2 = 0$, so our answer becomes $\boxed{20}$.

B: Set up an integral, which is the sum of infinitely many cylinders with radius x and height dx, where dx ranges from $\frac{A}{40}$ to 2. The integral becomes

$$\int_{\frac{A}{20}}^{2} \pi r^2 h = \int_{\frac{A}{20}}^{2} \pi\left(\sqrt{4 - x^2}\right)^2 dx = \int_{\frac{A}{20}}^{2} \pi(4 - x^2)dx.$$

This is equal to $\pi\left(4\left(2 - \frac{A}{20}\right) - \frac{2^3 - \left(\frac{A}{20}\right)^3}{3}\right)$. At $A = 20, \frac{A}{20} = 1$, so the answer becomes $\pi\left(4(2 - 1) - \frac{8-1}{3}\right) = \frac{5\pi}{3}$. Thus $B = \boxed{8}$.

C: Complete the square twice, and factor:

$$x^2 + 8x + 16 - (y^2 - 6y + 9) = B + 7,$$
$$(x + 4)^2 - (y - 3)^2 = B + 7,$$
$$(x + y + 1)(x - y + 7) = B + 7.$$

The only restrictions are that x and y are integers. That means for any ordered pair of the same parity that multiply to $B + 7$, there is an integer solution for (x, y). At $B = 8$, $B + 7 = 15$. This means there are $\boxed{8}$ solutions. Namely when 15 is split to $(1,15), (3, 5), (5, 3), (15, 1)$ and their negations.

Question 8

A: Split the hexagon into two isosceles trapezoids. Drop the altitudes from the vertexes of the shorter edge. This forms 2 45-45-90 triangles. This means that the height of the trapezoid is $\frac{\sqrt{2}}{2}$ and that the longer base has length $\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}=1+\sqrt{2}$. This means that the area of one of the trapezoids is $\frac{1}{2}\cdot\frac{\sqrt{2}}{2}\cdot\left(1+\sqrt{2}+1\right)$. Double this to get the answer of $1+\sqrt{2}$. The requested answer is $1+1=\boxed{2}$.

B: A yards is $3A$ feet. The work required to pull Kevin's weight up is $150\cdot 3A=450A$. Now, we can say that the rope is $\frac{3A}{2}$ feet under, with weight $2\cdot 3A=6A$. The work done to pull up the rope is then $\frac{3A}{2}\cdot 6A=9A^2$. This means that the total work done is $450A+9A^2$. At $A=2$, This equals $900+36=936$. The sum of the digits is $9+3+6=\boxed{18}$.

C: If Samuel, Nick, and Kevin take up 3 of the A parking spots, there are $k-3$ spots left to distribute among 4 regions, one to the left of the first car, one between the first two, one between the last two, and one after the last car. Call the number of cars in these regions a,b,c, and d, respectively. Then, we have

$$a+b+c+d=k-3.$$

However, b and c must be at least one, since they are between two parked cars. Let $b'=b-1$ and $c'=c-1$. Then, we have

$$a+b'+1+c'+1+d=k-3,$$
$$a+b'+c'+d=k-5,$$

Where a,b',c', and d are all nonnegative integers. This is a classic stars and bars question, with $k-5$ stars and 3 bars. There are $\binom{k-2}{3}$ ways to arrange these parking spaces to be in the different regions. Additionally, the three cars themselves can be arranged in $3!=6$ ways

At $B=18$, $k=9$, our final answer is $6\cdot\binom{7}{3}=\boxed{210}$.

Question 9

A: Let $y=2^x$. The equation becomes $y^3-y^2-8y-2y-8=0\rightarrow y^3-y^2-10y-8=0$, which factors as $(y+1)(y+2)(y-4)$. $y=-1$ and $y=-2$ are extraneous, but $y=4$ is a solution, at $x=\boxed{2}$.

B: $P_G = \frac{(N+A)(N+A-1)}{(2N+A)(2N+A-1)}$. $P_B = \frac{(N)(N-1)}{(2N+A)(2N+A-1)}$. $P_G - P_B = \frac{\left((N+A)^2-(N+A)-N^2+N\right)}{(2N+A)(2N+A-1)} = \frac{\left(2NA+A^2-A\right)}{(2N+A)(2N+A-1)} = \frac{(2N+A-1)(A)}{(2N+A)(2N+A-1)} = \frac{A}{2N+A}$. Since we want $\frac{A}{2N+A} < \frac{1}{13}$, given A and N are positive, this means that

$$A \cdot 13 < 2N + A,$$
$$12A < 2N,$$
$$N > 6A.$$

At $A = 2$, $N > 12$ so $N = \boxed{13}$ is the least possible value.

C: Notice that the x^2 coefficient is 0, which means that by Vieta's, $p + q + r = 0$. We can rewrite our expression as

$$\frac{r}{-r} + \frac{p}{-p} + \frac{q}{-q}.$$

Note that the value of B does not matter, our answer is simply $\boxed{-3}$.