1. D
2. C
3. C
4. C
5. C
6. A
7. C
8. A
9. B
10. D
11. C
12. B
13. B
14. C
15. C
16. A
17. B
18. A
19. D
20. B
21. A
22. A
23. C
24. B
25. C
26. A
27. E
28. A
29. B
30. D

1. D The sum of the geometric series is $\frac{1}{1-r}$ where $|r| < 1$ and $r \neq 0$. When r is negative, this produces the range $\left(\frac{1}{2}, 1\right)$. When r is positive, this produces the range $(1, \infty)$. The combined ranges are $\left(\frac{1}{2}, 1\right) \cup (1, \infty)$. $r = 0$ is not a geometric series.

2. C Let the first term equal x. Then the sum of the series is $\frac{x}{1-1/x} = \frac{x^2}{x-1}$. The derivative of this is $\frac{(x-2)x}{(x-1)^2}$. The sum has a relative minimum of 4 at $x = 2$, but the relative maximum at $x = 0$ is not a valid series since x must have an absolute value greater than 1. Thus, the smallest negative sum is when x approaches -1, where the sum approaches but is not equal to $-\frac{1}{2}$. $-\frac{1}{2} + 4 = \frac{7}{2}$.

3. C Let $a_{2n-1} = r^2$. Then $a_{2n} = rs$, $a_{2n+1} = s^2$, and $a_{2n+2} = 2s^2 - rs = s(2s - r)$. With the initial conditions given, this gives $a_{2n-1} = (n+1)^2$ and $a_{2n} = (n+1)(n+2)$. $\sum_{n=1}^{18} a_n = \sum_{n=2}^{10}\left(n^2 + n(n+1)\right) = \sum_{n=2}^{10}(2n^2 + n) = -3 + \sum_{n=1}^{10}(2n^2 + n) = -3 + \frac{10\cdot 11\cdot 21}{3} + \frac{10\cdot 11}{2} = -3 + 770 + 55 = 822$.

4. C $h_n = \frac{n}{1+2^{-1}+2^{-2}+\cdots+2^{-n}}$. The denominator is a geometric series whose sum approaches 2 as n grows very large, so $h_n \sim \frac{n}{2}$.

5. C The equation will eventually stabilize, since $\frac{9}{4} < 3$. Solving $x = \frac{9}{4}x(1-x)$ gives $1 - x = \frac{4}{9}$ and $x = \frac{5}{9}$.

6. A $\lim\limits_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim\limits_{x\to 0} \frac{\sin x}{x} = 1$, so by the Limit Comparison Test, the series is equivalent to the Basel problem and is thus absolutely convergent.

7. C $\sqrt{n+1} \approx \sqrt{n}$ and $\frac{1}{n+2} \approx \frac{1}{n}$ as n grows large, so the series is asymptotic to $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ and is thus conditionally convergent.

8. A From the given values, $\tan x = 1 \cdot x + \frac{2}{6} \cdot x^3 + \frac{16}{120} \cdot x^5 + \cdots = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$. The approximation for $\tan 1$ using the degree-5 Maclaurin series is $1 + \frac{1}{3} + \frac{2}{15} = \frac{22}{15}$, so $15 \tan 1 \approx 22$.

9. B Note that for $f(x) = px^{4n} + qx^{4n-1} + rx^{4n-2} + sx^{4n-3}$, $f(1) = p + q + r + s$, $f(i) = p - qi - r + si$, $f(-1) = p - q + r - s$, and $f(-i) = p + qi - r - si$ so $r = \frac{f(1)+f(-1)-f(i)-f(-i)}{4}$. For the given function, obviously $f(1) = f(i) = 1$. $f(-1) = (-2-i)^8 = (3+4i)^4 = (-7+24i)^2 = -527 - 336i$. Similarly, $f(-i) = (-1-2i)^8 = (-3+4i)^4 = (-7-24i)^2 = -527 + 336i$. Evaluating the expression gives $-168i$. The magnitude of this is 168. For the last step in both of those calculations, you may also use the fact that $(a+bi)^2 - (a - bi^2) = 4abi$ or even $(a+bi)^4 - (-a+bi)^4 = 8abi(a^2 - b^2)$.

10. D When $x > 0$, this is the power series of $\cos\sqrt{x}$, so the radius of convergence is ∞. Note that when $x < 0$, the function is $\cosh\sqrt{|x|}$.

11. C The Maclaurin series of the denominator is quartic (with coefficient $-\frac{1}{2}$). Polynomial multiplication gives the Maclaurin series of $\ln^2(1-x)$ as $x^2+x^3+\frac{11x^4}{12}+O(x^5)$, so the quartic coefficient of the numerator is $-\frac{11}{12}$ and the limit is $\frac{11}{6}$. $11+6=17$.

12. B The first three derivatives of $\sqrt{1+x}$ are $\frac{1}{2(1+x)^{1/2}}$, $-\frac{1}{4(1+x)^{3/2}}$, and $\frac{3}{8(1+x)^{5/2}}$. Evaluated at $x=0$, the third derivative is $\frac{3}{8}$, so the associated coefficient is $\frac{3/8}{3!}=\frac{1}{16}$.

13. B Note that this is the integral of $\tan x$, whose Maclaurin series is $x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots$ from question 8. Integrating this gives the x^6 term of the Maclaurin series of $\ln(\sec x)$ as $\frac{1}{45}$.

14. C $n^n \le \sum_{m=1}^{n} m^m \le n\cdot n^n$. Dividing by n^n and taking the nth root, we have $1 \le \frac{1}{n}\left(\sum_{m=1}^{n} m^m\right)^{\frac{1}{n}} \le n^{\frac{1}{n}}$. By the Squeeze Theorem, the sum limits to 1 as n approaches ∞. Using Stirling's approximation, $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, the desired limit equals e.

15. C By partial fractions, this is equal to $\frac{1}{x-3}-\frac{1}{x-2}=\frac{1/2}{1-\frac{x}{2}}-\frac{1/3}{1-\frac{x}{3}}$. Converting these to geometric series gives $\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots\right)-\frac{1}{3}\left(1+\frac{x}{3}+\frac{x^2}{9}+\frac{x^3}{27}+\cdots\right)$. The numerator of the coefficient of the x^3 term is $81-16=65$.

16. A This is the number of solutions in positive integers to the Diophantine equation $a+2b=100$, or the number of non-negative integer solutions to $a'+2b'=97$. a' must be odd, but there are no other constraints. This makes for 49 sequences with $a' \in \{1,3,5,7,\ldots,97\}$.

17. B Note that $45^2-45=2025-45=1980$, so the infinite nested radical is 45.

18. A This is equivalent to the sum of the first 100 positive integers. $\frac{100\cdot 101}{2}=5050$.

19. D Let the solutions be $a-b$, a, and $a+b$. The sum of the solutions is $3a$, the sum of the solutions taken two at a time is $(a-b)a+a(a+b)+(a+b)(a-b)=3a^2-b^2$, and the product of the solutions is $a(a^2-b^2)$. By Vieta's, $3a=15\sqrt{3}$ so $a=5\sqrt{3}$ and $a^2=75$. $a^2-b^2=69$ so $b^2=6$. $3a^2-b^2=225-6=219$.

20. B The Taylor series for sine has an infinite radius of convergence. $\sin 2024\pi = 0$.

21. A The sum of the terms in $\{a, a+b, a+2b, \ldots, a+10b\}$ is $11a+55b$. Setting this equal to 2024 and dividing by 11 gives $a+5b=184$. a must be 4 more than a multiple of 5. If $a=5n+4$, then $b=36-n$. n can range from 0 to 35 for a total of 36 solutions.

22. A $1+n$ is asymptotic to n as n grows large. Let $L=\sqrt[2n]{n}$. Then $\ln L=\frac{\ln n}{2n}$, which tends to 0 as n grows large, meaning L tends to 1. Note that $\lim_{n\to\mathbf{0}} \sqrt[2n]{1+n}=\sqrt{e}$.

23. C The sum is equal to $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{1/n}{1+(1+k/n)^2}=\int_0^1\frac{dx}{1+(1+x)^2}=\int_1^2\frac{du}{1+u^2}=\arctan 2-\frac{\pi}{4}$. The tangent of this is $\frac{2-1}{1+2\cdot 1}=\frac{1}{3}$.

24. B By the Product Rule, the indefinite integral is $\prod_{i=1}^{2024}(x+i)$, which evaluated at the bounds is $2025!-2024!=2024\cdot 2024!$, which divided by $2025!$ is $\frac{2024}{2025}$.

25. C The derivative of $\frac{8n^2}{n^3+512}$ is $-\frac{8n(n^3-1024)}{(n^3+512)^2}$, and so the continuous function has a local maximum when $n^3 = 1024$, or $n = 8\sqrt[3]{2}$. Note the extreme proximity of 1024 to 1000 (compared to 1331), which would correspond to a maximum at $n = 10$ in the discrete sequence.

26. A $(x+4)^2 < 9$ and the interval of convergence is $(-7, -1)$.

27. E The summand equals $\frac{(-i)^n}{n!}$, so the sum is $e^{-i} = e^{i\cdot(-1)} = \cos(-1) + i\sin(-1) = \cos 1 - i\sin 1 = -is + c$.

28. A Note that $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{e^x - 1}{x}$, ignoring the constant of integration. Evaluated at 1, this is $e - 1$. The limit taken as x approaches 0 equals $\lim_{x\to 0} \frac{e^x}{1} = 1$ by l'Hospital's, so the integral is $e - 2$.

29. B Let $y = \frac{x}{x + \frac{x}{x+\cdots}}$. To find the inverse, swap variables so $x = \frac{y}{y + \frac{y}{y+\cdots}}$ or $x = \frac{y}{y+x}$. Solving for y gives $y = \frac{x^2}{1-x} = x^2 + x^3 + x^4 + \cdots$. This is all but $1 + x$ of the second term. Since the continued fraction and part of the infinite series are inverses of each other and $y\left(\frac{1}{2}\right) = \frac{1}{2}$, the sum of their integrals is the area of the square $\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$, which is $\frac{1}{4}$. Adding the remaining $\int_0^{1/2} (1+x)\,dx = \frac{5}{8}$ gives a total sum of $\frac{7}{8}$.

30. D The evolution of the worm is as follows, where $0\{n\}$ represents n copies of 0.

Gen.	Worm
0	(2)
1	(1 1)
2	1 0 1 0 1 0
3	1 0 1 0 (1)
4	1 0 1 0 0 0 0 0 0
5	1 0 1 0 0 0 0 0
...	...
10	1 0 (1)
11	1 0{13}
12	1 0{12}
...	...
24	(1)
25	0{26}
26	0{25}
...	...
50	0
51	[Lifetime = 51]