1. D
2. C
3. C
4. C
5. B
6. D
7. A
8. C
9. C
10. B
11. B
12. C
13. D
14. D
15. A
16. E
17. A
18. B
19. D
20. A
21. D
22. B
23. B
24. A
25. A
26. B
27. C
28. B
29. E
30. A

1. D The word has 10 letters. Looking at the repeated letters, we see there are 2 Rs and 2 Fs. This gives us $\frac{10!}{2!*2!} = \frac{10!}{2^2}$ total permutations. It's clear that m can't equal 1, so we have $n = 10$ and $m = 2$, giving us an answer of 12.
2. C Given any choice of 6 different digits from 0 to 9, there is exactly one ordering for which they are strictly decreasing. Thus the answer is simply the number of ways to choose 6 digits out of 10 possibilities, which is $\binom{10}{6} = 210$.
3. C We can think of the cards as arranged in a line where the first 13 cards represent the first player's cards, the second 13 represent the second players cards and so on. There are 52! total arrangements of the cards. All the spades must be in slots 1-13, 14-26, 27-39, or 40-52, which give us 4 possibilities corresponding to the 4 players, there are 13! ways to arrange the spades and 39! ways to arrange the other cards. Thus the desired probability is $\frac{4*39!*13!}{52!} = \frac{4}{\binom{52}{13}}$.
4. C We can think of this geometrically. $a + bi$ lies in a $12\ x\ 12$ square in the complex plane centered around 0 with side parallel to the axes. For the number to have magnitude less than $4\sqrt{3}$, it must lie inside the circle centered at the origin with radius $4\sqrt{3}$, but it also must lie in the $12\ x\ 12$ square. Thus the desired probability is the area of intersection between the circle and square divided by the area of the square, which is 144. The four sections of the circle that are outside the square have areas that can be calculated as the difference between the area of a 60 degree circular sector, which is 8π, and the area of an equilateral triangle with side length $4\sqrt{3}$, which is $12\sqrt{3}$. Thus the area of intersection is $48\pi - 4(8\pi - 12\sqrt{3}) = 16\pi + 48\sqrt{3}$, meaning the desired probability is $\frac{16\pi+48\sqrt{3}}{144} = \frac{\pi+3\sqrt{3}}{9}$.
5. B There are 7! = 5040 total distinguishable ways for 8 people to stand in a circle. There are $2 * 6! = 1440$ ways for Jeremy and Julian to stand next to each other since there are 2 orders in which they could stand and 6! ways to arrange the other 6 people. Thus there are $5040 - 1440 = 3600$ valid arrangements.
6. D $w = 0.7$, which would occur in the case that B happens every time that A happens. $y = 1$, since $P(A) + P(B) > 1$. $x = 0.6$, which we can determine using the equation $P(A) + P(B) - P(A \cup B) = P(A \cap B)$ where $P(A \cup B) = 1$. $z = 0.9$, which occurs in the same case that $w = 0.7$. Thus we have $0.7 + 1 + 0.6 + 0.9 = 3.2$.
7. A The slope of the normal line at $x = a$ is $-a$. Thus we want the expected value of $-x$ on the interval $[1, 3]$ which is simply -2.
8. C This region is a sphere with radius 1, meaning it has volume $\frac{4\pi}{3}$. The locus of points inside the sphere with a y-coordinate greater than $\frac{1}{2}$ is a spherical cap whose volume can be found by rotating the graph of $0 \le y \le \sqrt{1 - x^2}$ around the x-axis from $x = \frac{1}{2}$ to $x = 1$. Using the disk method, we find this volume to be $\frac{5\pi}{24}$. Thus the answer is the ratio of these two volumes, which is $\frac{5}{32}$.

9. C The first graph is a circle with radius 2, meaning it has area 4π. The second graph is a lemniscate, half of which is inside the circle. The area of the lemniscate is 1, so the area of half the lemniscate is $\frac{1}{2}$. The answer is the ratio of the area of the part of the lemniscate that's inside the circle to the area of the circle, which is $\frac{1}{8\pi}$.

10. B The distribution of the number of flips to get to the first tails is a geometric distribution with $p = \frac{1}{2}$. The expected value of a geometric distribution is $\frac{1}{p}$, so the answer is 2.

11. B The number of natural numbers less than 72 that are relatively prime to 72 is given by Totient function, which is calculated as $72 * \frac{1}{2} * \frac{2}{3} = 24$. Because $216 = 3 * 72$ there are also $3 * 24 = 72$ numbers in the range from 1 to 216 that are relatively prime to 72. Thus, the desired probability is $\frac{72}{216} = \frac{1}{3}$.

12. C We first find the probability of rolling each face. Calling the probability of rolling a one p, we know the probabilities of one through eleven are

$$p, 2p, 3p, 4p, 5p, 6p, 5p, 4p, 3p, 2p, p.$$

The sum of these probabilities should equal 1. So, we find $36p = 1$ or $p = \frac{1}{36}$. Now, for the problem itself. It is equivalent to asking the probability that at least one face is a multiple of three. Thus we will take the probability that no die comes up as a multiple of 3 and take the complement. The probability that a die does not come out as a multiple of 3 is $1 - \frac{3}{36} - \frac{6}{36} - \frac{3}{36} = \frac{2}{3}$. Thus the desired probability is $1 - \left(\frac{2}{3}\right)^{31}$.

13. D Rolling one of these 11-sided dice is equivalent to rolling two fair six-sided dice and subtracting 1 from their sum. Thus the sum of the 31 dice has the same probability distribution as the sum of 62 fair 6-sided dice minus 31.

Since we subtract 31 from the total sum of our 6-sided dice rolls, we want the sum of the rolls to be equivalent to $31 \equiv 1 \ (mod\ 3)$. Note that no matter what the first 61 fair six-sided dice rolls are, the 62nd die always has a $\frac{1}{3}$ chance of making the sum 1 modulo 3, since there are always two rolls that work. Thus the answer is $\frac{1}{3}$.

14. D Let k be the expected value of the result. If an R is rolled, the new die has an expected result of $2k$. Thus, we can solve for k as $k = \frac{1}{6} * 1 + \frac{1}{6} * 2 + \frac{1}{6} * 3 + \frac{1}{6} * 4 + \frac{1}{6} * 5 + \frac{1}{6} * (2k)$, thus $k = 3.75$.

15. A We seek the number of derangements of 10000 divided by 10000!. The number of derangements of n is approximately $\frac{n!}{e}$, so the desired probability is $\frac{1}{e}$.

16. E Solving the cubic, we get solutions of $\frac{2}{3}$, -1, and $\frac{-6}{5}$. The only value that is a valid probability is $\frac{2}{3}$, which gives us an answer of 5.

17. A Given the result of the first button press, because the blue button has a $\frac{n}{100}$ probability of turning the light on and the red button has a probability of 1 of turning the light on, the probability the blue button was selected is $\frac{\frac{n}{100}}{1+\frac{n}{100}} = \frac{3}{8}$. Solving this gives us $n = 60$. Given the result of the second button press, the blue button has a

probability of $\frac{9}{25}$ of turning the light on twice while the red button has a probability of 1 of doing so. Thus, the probability that the blue button was selected is $\frac{\frac{9}{25}}{1+\frac{9}{25}} = \frac{9}{34}$.

18. B Taking the sum $\sum_{k=0}^{\infty} \frac{n*2^k}{k!}$, we get $n * e^2$. Since the sum of all the probabilities must be 1, we have $n = \frac{1}{e^2}$.

19. D We will use complementary counting. (X, Y) cannot be $(0,0), (1,0), (0,1), (1,1), (2,0)$ or $(0,2)$ which have probabilities of $n^2, 2n^2, 2n^2, 4n^2, 2n^2$, and $2n^2$ for a sum of $13n^2$. Thus the desired probability is $1 - 13n^2$.

20. A The height of the region is k, so the width will be $2\sqrt{k}$, making the area $\frac{4}{3}k\sqrt{k}$. In a symmetric parabolic sector, the distance from the base to the centroid is $\frac{2}{5}$ the height. So, by Pappus' Theorem, the volume of the solid is $2\pi * \frac{2}{5}k * \frac{4}{3}k\sqrt{k} = \frac{16}{15}\pi k^{\frac{5}{2}}$. The expected value of $\frac{16}{15}\pi k^{\frac{5}{2}}$ can be found using the integral

$$\frac{16}{15}\pi \int_1^5 \frac{1}{4} * k^{\frac{5}{2}}\, dk = \frac{1000\sqrt{5} - 8}{105}\pi.$$

Thus the answer is 1118.

21. D The constant term is $\binom{15}{6}(x^3)^6(\frac{-1}{x^2})^9 = -5005$.

22. B We can build a recursion. Let a_n be the number of ways to tile a $1\ x\ n$ board, where $n \geq 3$. There are 2 ways to put a $1\ x\ 1$ tile at the end, leaving a_{n-1} ways to tile the remaining squares. There are 8 ways to put a $1\ x\ 2$ tile at the end, leaving a_{n-2} ways to tile the remaining squares. Thus the recursion is $a_n = 2a_{n-1} + 8a_{n-2}$. We can find that $a_1 = 2$ and $a_2 = 12$. Building up the recursion we find that $a_3 = 40, a_4 = 176, a_5 = 672$, and $a_6 = 2752$, which is our answer.

23. B We will find an explicit formula that corresponds to this recursion. The characteristic polynomial is $x^2 - 2x - 8 = 0$, which has roots 4 and -2. Thus, our formula is in the form $a * 4^n + b * (-2)^n$. We know that $a_1 = 2$ and $a_2 = 12$, so we have $4a - 2b = 2$ and $16a + 4b = 12$, which gives us $a = \frac{2}{3}$ and $b = \frac{1}{3}$. Thus, we know

$$a_n = \frac{2}{3}(4)^n + \frac{1}{3}(-2)^n.$$

As n approaches infinity, $(-2)^n$ becomes negligible relative to 4^n, so our answer is simply the value of a which is $\frac{2}{3}$.

24. A By the Principle of Inclusion and Exclusion, $|A \cup B \cup C \cup D| = a_1 - a_2 + a_3 - a_4$. We are given everything except for a_1, so we plug in the values and solve for a_1 to get $a_1 = 2020$.

25. A Using the Principle of Inclusion and Exclusion once again, a_2 counts all the triples ($|A \cap B \cap C|, |A \cap B \cap D|$, etc.) three times, so we must subtract a_3 twice. a_2 counts a_4 6 times, and a_3 counts a_4 4 times, so a_4 is already counted $6 - 2 * 4 = -2$ times. Therefore, we must add a_4 3 times. Thus the expression for our answer is $a_2 - 2 * a_3 + 3 * a_4$. Plugging in the values we are given, we find that our answer is $990 - 2 * 180 + 3 * 10 = 660$.

26. B There are 5 S, so there are 6 possible positions of the P with respect to the 5 S. In four of these positions, there is at least one S on either side of the P, so $\frac{4}{6}$ of the arrangements satisfy the condition.

27. C The median is the value of a such that $\int_0^a e^{-x}dx = \frac{1}{2}$. This gives us $1 - e^{-a} = \frac{1}{2}$, which means $a = ln(2)$.

28. B The variance of a random variable is $E[X^2] - E[X]^2$. $E[X^2] = \int_0^\infty x^2e^{-x}dx$ and $E[X] = \int_0^\infty xe^{-x}dx$. Thus $f(x) = x^2e^{-x}$ and $g(x) = xe^{-x}$.

29. E This is an exponential distribution and all four of these are properties of the exponential distribution.

30. A A probability must be greater than or equal to 0, meaning it must be strictly greater than -1.