Mu Ciphering Nationals 2024 solutions

0. $y' = \cos x + 1 \to y' = 2 \to y = 2x$

1. $\pi\int_0^2 [\left(4x - x^2\right)^2 - x^4]dx = \frac{32\pi}{3}$

$2\pi\int_0^2 (3 - x)\left(4x - 2x^2\right)dx = \frac{32\pi}{3} \to \frac{64\pi}{3}$

2. Draw a good 2-dimensional picture. You create a kite when you connect the centers to the intersection points. Diagonals of a kite are perpendicular. Call the piece you want x and the other diagonal call the pieces y and 8-y.

$6^2 - \left(8 - y\right)^2 = 4^2 - y^2 \to 16y = 44 \to y = \frac{11}{4}$

$16 - \frac{121}{16} = x^2 \to x = \frac{3\sqrt{15}}{4}$

3. $V = \frac{1}{3}\pi r^2 h \to h = \frac{4r}{3} \to l = \frac{5r}{3} \to V = \frac{4\pi r^3}{9}$

$SA = \pi r^2 + \pi r l = \frac{8\pi r^2}{3} \to \frac{dV}{dt} = \frac{4\pi r^2}{3}\frac{dr}{dt} = -2 \to \frac{dr}{dt} = \frac{-1}{24\pi}$

$\frac{dSA}{dt} = \frac{16\pi r}{3}\frac{dr}{dt} = \frac{-4}{3} \to \frac{4}{3}$

4. A good picture and some 30-60-90 triangles will get this done. Draw UE. That creates a 30-30-120 triangle. Draw altitudes from O and H down to UE. This creates 30-60-90's with 18 as the hypotenuse. Therefore, the latitudes you drew are $9\sqrt{3}$. So, UE equals 9+12+9=30 and US is $10\sqrt{3}$

5. $$\frac{r\cos\theta+\frac{dr}{d\theta}\sin\theta}{-r\sin\theta+\frac{dr}{d\theta}\cos\theta}\rightarrow\frac{dr}{d\theta}=-6\cos 2\theta\rightarrow\frac{\frac{-3\sqrt{3}}{2}\bullet\frac{\sqrt{3}}{2}-3\bullet\frac{1}{2}}{\frac{3\sqrt{3}}{2}\bullet\frac{1}{2}-3\bullet\frac{\sqrt{3}}{2}}$$
$$\frac{\frac{-9}{4}-\frac{3}{2}}{\frac{3\sqrt{3}}{4}-\frac{3\sqrt{3}}{2}}=\frac{\frac{-15}{4}}{\frac{-3\sqrt{3}}{4}}=\frac{5\sqrt{3}}{3}$$

6. Draw yourself a 6x6 to get a visual (9,12), (10,11),(10,12),(11,12) also (11,11) and (12,12)

$$\frac{2(4\bullet1+3\bullet2+3\bullet1+2\bullet1)+2\bullet2+1\bullet1}{36\bullet36}=\frac{35}{1296}$$

7. $$f(x)=Ax^3+Bx^2+Cx+D\rightarrow f'(x)=3Ax^2+2Bx+C$$
$$f''(x)=6Ax+2B\rightarrow12A+2B=0\rightarrow B=-6A\rightarrow3A-2B+C=0$$
$$C=-15A\rightarrow8A+4B+2C+D=-46A+D=-6\rightarrow-A+B-C+D=8A+D=48$$

A=1, D=40, B=-6, and C=-15 so answer is 20

8. This gets a lot easier if you remember your sum of cubes factoring. It cleans to $\left(2k+\frac{3}{k}\right)^3$. It is very friendly from here and you can use multiple routes. $k=\frac{\sqrt{6}}{2}$ plug in to get $48\sqrt{6}$

9. $$\int_{-5}^{2}\left|x^3-2x^2-9x+18\right|dx=\int_{-5}^{-3}-\left(x^3-2x^2-9x+18\right)dx+\int_{-3}^{2}\left(x^3-2x^2-9x+18\right)dx$$
$$\frac{-x^4}{4}+\frac{2}{3}x^3+\frac{9}{2}x^2-18x\Big|_{-5}^{-3}+\frac{x^4}{4}-\frac{2}{3}x^3-\frac{9}{2}x^2+18x\Big|_{-3}^{2}=\frac{280}{3}+\frac{875}{12}=\frac{665}{4}$$

10.
$$rs=\sqrt{s(s-w)(s-i)(s-g)}\rightarrow r(121+x)=\sqrt{(121+x)x\bullet 101\bullet 20}$$
$$r^2(121+x)=2020x\rightarrow r^2=\frac{2020x}{x+121}\rightarrow \lim_{x\to\infty}\frac{2020x}{x+121}=2020\rightarrow r=\sqrt{2020}\rightarrow r=44$$
*Note that the function is increasing when $x \geq 0$.

11.
$$\int_0^{\frac{\pi}{4}}\frac{\sin x}{1+\cos-\sin^2 x}dx=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos+\cos^2 x}dx\rightarrow u=\cos x\rightarrow \int_1^{\frac{\sqrt{2}}{2}}\frac{-1}{u(u+1)}du=\int_1^{\frac{\sqrt{2}}{2}}\left(\frac{1}{u+1}-\frac{1}{u}\right)du$$
$$\ln\left(1+\frac{1}{u}\right)\Bigg|_1^{\frac{\sqrt{2}}{2}}=\ln\left(\frac{1+\sqrt{2}}{2}\right)$$

12. $\frac{1}{A}+\frac{1}{B}+\frac{1}{AB}=\frac{3}{10}\rightarrow 10A+10B+10=3AB\rightarrow 3AB-10A-10B=10$
$$A(3B-10)-\frac{10}{3}(3B-10)=\frac{130}{3}\rightarrow(3A-10)(3B-10)=130$$
$65\bullet 2, 26\bullet 5\rightarrow(12,5),(25,4),(5,12),(4,25)\rightarrow 12+25+5+4=46$

Answers:

0. y=2x

1. $\frac{64\pi}{3}$

2. $\frac{3\sqrt{15}}{4}$

3. $\frac{4}{3}$

4. $10\sqrt{3}$

5. $\frac{5\sqrt{3}}{3}$

6. $\frac{35}{1296}$

7. 20

8. $48\sqrt{6}$

9. $\frac{665}{4}$

10. 44

11. $\ln\left(\frac{1+\sqrt{2}}{2}\right)$

12. 46