Mu Bowl Solutions

<table>
<tr><th>#</th><th>Answer</th><th>Solution</th></tr>
<tr><td>0</td><td>3717</td><td>From 1 to 9, $\{\log(x)\} = \log(x)$
From 10 to 99, $\{\log(x)\} = \log\left(\frac{x}{10}\right)$
From 100 to 999, $\{\log(x)\} = \log\left(\frac{x}{100}\right)$
From 1000 to 9999, $\{\log(x)\} = \log\left(\frac{x}{1000}\right)$
Because $\log(x)$ is increasing, ordering $\log(x)$ is equal to ordering x.

A:
For these questions, we will count how many ‘n’ digit numbers are less than the value given
1-9: 1-2 -> 2
10-99: 10-23 -> 14
100 – 999: 100-239 -> 140
1000-9999: 1000 – 2399 -> 1400
1400+140+14+2=1556
$A: 1557$
B:
1-9: 1-2 ->2
10-99: 10-20 -> 11
100-999: 100-202 -> 103
1000-9999: 1000-2023 -> 1024
2+11+103+1024 = 1140
$B: 1141$
C:
the first few numbers are
1, 10, 100, 1000, 1001 – 1009, 101, 1010, 1011-1019: is the 24th number
$C: 1019$
Final:
$$1557 + 1141 + 1019 = 3717$$</td></tr>
<tr><td>1</td><td>$-\pi$</td><td>A:
$$\frac{x((4x^2+1)-(4x^2-1))}{\sqrt{4x^2-1}+\sqrt{4x^2+1}} = \frac{2x}{2x+2x} = \frac{1}{2}$$
B:
$$\ln(1-x) \sim -x$$
$$\ln(\cos(2x)) = \ln(1-\sin^2(2x))$$
$$\ln(\cos(x)) = \ln(1-\cos^2(x))$$
$$\frac{\ln(\cos(2x))}{\ln(\cos(x))} = \frac{-\sin^2(2x)}{-\sin^2(x)} = 4$$</td></tr>
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		C: $\lim_{x\to\infty} x\sin(x)$ could range anywhere from 0 to infinity depending on the value of x. Thus, the limit does not exist and $C=-1$. D: $$\lim_{n\to\infty}\sum_{i=1}^{n-1}\frac{\sqrt{n^2-i^2}}{n^2}=\frac{1}{n}\cdot\sqrt{1-\left(\frac{i}{n}\right)^2}=\int_0^1\sqrt{1-x^2}=\frac{\pi}{4}$$ E: $$-1\le\sin(x)\le 1$$ $$\lim_{x\to\infty}\frac{2x^2-3}{x^2}=2\le\lim_{x\to\infty}\frac{2x^2-3\sin(x)}{x^2}\le\lim_{x\to\infty}\frac{2x^2+3}{x^2}=2$$ Thus, $E=2$. Final: $$\frac{1}{2}\cdot 4\cdot(-1)\cdot\frac{\pi}{4}\cdot 2=-\pi$$
2	-154	A: $$(r_1+r_2)(r_3+r_4)+(r_1+r_3)(r_2+r_4)+(r_1+r_4)(r_2+r_3)$$ $$=2(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4)=2\cdot 1=2$$ B: $$r_1+r_2+r_3+r_4=4\rightarrow$$ $$(4-r_1)(4-r_2)(4-r_3)(4-r_4)=f(4)=256-256+16+20-1=35$$ C: $$f(x)=(x-r_1)(x-r_2)(x-r_3)(x-r_4)$$ $$\ln(f(x))=\ln(x-r_1)+\ln(x-r_2)+\ln(x-r_3)+\ln(x-r_4)$$ $$\frac{f'(x)}{f(x)}=\frac{1}{x-r_1}+\frac{1}{x-r_2}+\frac{1}{x-r_3}+\frac{1}{x-r_4}$$ $$\frac{f'(4)}{f(4)}=\frac{256-192+8+5}{256-256+16+20-1}=\frac{77}{35}=\frac{11}{5}$$ D: A root that should be tested with the Rational Root Theorem is -1. $$f(-1)=1+4+1-5-1=0$$ So -1 is the negative root. Final: (note that $BC=f'(4)$) $$2\cdot 35\cdot\frac{11}{5}\cdot(-1)=-154$$
3	734	$$N=2^3\cdot 3\cdot 5\cdot 6=2^4\cdot 3^2\cdot 5$$ A: $$(4+1)(2+1)(1+1)=30$$

		B: $lcm(2,3,5) = 30$ $$\phi(30) = 30 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 8$$ $$2024 = 30 \cdot 67 + 14$$ From $1 - 14$: $1,7,11,13,(4)$ numbers are relatively prime to 30. $$67 * 8 + 4 = 540$$ C: $$\frac{(2^2 + 2^3 + 2^4)(3 + 3^2)(1 + 5)}{(2 + 2^2 + 2^3 + 2^4)(3 + 3^2)(1 + 5)} = \frac{14}{15}$$ $$14 + 15 = 29$$ D: Let $v_p(n)$ be the maximum power of p that divides n $$-4 \leq v_2(n) \leq 4$$ $$-2 \leq v_3(n) \leq 2$$ $$-1 \leq v_5(n) \leq 1$$ There are $9 \cdot 5 \cdot 3 = 135$ possibilities for $\frac{d_1}{d_2}$. Final: $$30 + 540 + 29 + 135 = 734$$
4	619	A: $$\frac{2}{3}(4)(2^2) = \frac{32}{3} \rightarrow 35$$ B: $$-x^2 + 10x - 21 = 4 - (x - 5)^2$$ $$\frac{1}{2}\big(y(3) + 2y(4) + 2y(5) + 2y(6) + y(7)\big) = 16 - \frac{1}{2}(4 + 2 + 0 + 2 + 4)$$ $$= 10$$ C: The point with the minimum distance will have a tangent line parallel to $y = x$ $$-2a + 10 = 1 \rightarrow a = \frac{9}{2}$$ Point: $\left(\frac{9}{2}, \frac{15}{4}\right)$ Applying point to line, $$\frac{\frac{9}{2} - \frac{15}{4}}{\sqrt{2}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8} \rightarrow 13$$ D: Centroid of R: $$\left(5, \frac{2}{5} \cdot 4\right) = \left(5, \frac{8}{5}\right)$$ $$2\pi RA = 2\pi \cdot \frac{5 - \frac{8}{5}}{\sqrt{2}} \cdot \frac{32}{3} = \pi \cdot \frac{17}{5} \cdot 32\frac{\sqrt{2}}{3} = \frac{544}{15}\sqrt{2} \rightarrow 561$$ Final:

		$35 + 10 + 13 + 561 = 619$
5	4	Using continuity for both points and differentiability for the first, $a + b + 2c = 3 \quad E_1$ $3a + b + 4c = 12 \quad E_2$ $a - 4b - 12c = 6 \quad E_3$ $E_2 - E_1$ gives $A + C = \frac{9}{2}$. $E_2 - 3E_1$ gives $B + C = -\frac{3}{2}$. $E_1 - E_3$ gives $5B + 14C = -3$, so $3B + 12C = 0$ and $B = -4C$. Thus, $B = -2$ and $C = \frac{1}{2}$, so $A = 4$. D: The only issue with differentiability happens when the arguments of the absolute value is 0, and when it isn't, it does not affect the differentiability. Thus, $\frac{d}{dx}[h(x)]$ at 0^+ = $\frac{d}{dx}[h(x)]$ at 0^- At 0^+ $h(x) = Dx + \arctan(x), h'(0) = D + 1$ At 0^- $h(x) = -Dx - \arctan(x), h'(0) = -D - 1$ Since $D + 1 = -D - 1$ $D = -1$ Final: $4 \cdot (-2) \cdot \frac{1}{2} \cdot (-1) = 4$
6	21	We can write $f(x)$ as an infinite geometric series with common ratio $2x$ and first term $1 + x$. $f(x) = (1 + x) + 2x(1 + x) + 4x^2(1 + x) + 8x^3(1 + x) + \cdots = 1 + 3x + 6x^2 + 12x^3 + 24x^4 + \cdots 3(2)^{n-1}x^n$. A: $f'(0) = 3 + 12(0) + 36(0)^2 + \cdots = 3$ B: $f''(0) = 12 + 36(0)^2 + 96(0)^3 + \cdots = 12$ C: From the Maclaurin series, $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, $g(x) = x - \frac{x^2}{3} + \frac{x^3}{5} - \frac{x^4}{7} + \cdots$ and $g'(x) = 1 - \frac{2x}{3} + \frac{3x^2}{5} - \frac{4x^3}{7} + \cdots$. $g(0^+) = 1$. D: Continuing to differentiate, $g''(x) = -\frac{2}{3} + \frac{6x}{5} - \frac{12x^2}{7} + \cdots$. $g''(0^+) = -\frac{2}{3}$. $2 + 3 = 5$. Final: $3 + 12 + 1 + 5 = 21$
7	306	A: $(1 + 1 + 2 - 3)^8 = 1$ B: $11\ C\ 3 = 165$

		C: Due to symmetry, number of terms $\deg(x) > \deg(y)$ is equal to number of terms $\deg(y) > \deg(x)$ Number of terms of $\deg(x) = \deg(y) \rightarrow$ $$9+7+5+3+1=25$$ $$\frac{165-25}{2}=70$$ D: $\deg(z)$ is odd $$9C2+7C2+5C2+3C2=36+21+10+3=70$$ Final: $$1+165+70+70=306$$
8	-400	A: $$a(x)=(x-50)^2-476$$ So $a(x)$ achieves a minimum when $x=50$ at $y=0-476=-476$. B: $$b'(x)=\frac{2(x^2+1)-2x\cdot 2x}{(x^2+1)^2}=\frac{2-2x^2}{(x^2+1)^2}$$ So $b(x)$ achieves a minimum when $x=-1$ at $y=-\frac{2}{2}=-1$. C: $$c'(x)=3x^2-6x-9=3(x-3)(x+1)$$ So $c(x)$ has a local minimum when $x=3$ at $y=27-27-27+4=-23$. The other candidate point is when $x=-2$, where $y=-8-12+18+4=2$. There are no other candidate points because by graphing the cubic, $c(5)>c(3)$ and $c(-1)$ is a local maximum. So $c(x)$ achieves a minimum when $x=3$ at $y=-23$. D: For every interval $[n, n+1]$, the function is a line segment. since those are monotonic, the minimum must be in a vertex. Now, we can see the change of each $\|x-k\|$ when x increases by 1. If $x \geq k, \|x-k\|$ increases by 1 when x increases else, $\|x-k\|$ decreases by 1 When x goes from k to $k+1$ $\|x-1\|$ through $\|x-k-1\|$, $(k-1)$ terms increase and the rest decreases We want to stop adding when change is positive, $$(k-1)-(20-(k-1))<0$$ $$2k-22<0$$ $$k=10$$ *choosing k=11 works as well $$9+8+7+\cdots+0+1+2+10=100$$

		Final: $$-476-1-23+100=-400$$
9	169	A: Due to the symmetry of a circle, $H+h+K+k$ is twice the sum of the coordinates of the center of the circle. $$2\cdot(3+4)=14$$ B: Point P lies on the graph, so $(a-3)^2+(b-4)^2=4$. Point P also lies on $y=x$, so $a=b\rightarrow(a-3)^2+(a-4)^2=4\rightarrow 2a^2-14a+21=0$. Through graphing, $y=x$ intersects the circle at two points, meaning the two roots of the quadratic are the two distinct values of a. They sum to $-\frac{-14}{2}=7$ by Vieta's. $$B:7$$ C: Note that a^2+b^2 is effectively the distance from the origin squared. min: $\left(\sqrt{3^2+4^2}-2\right)^2=9$ max: $\left(\sqrt{3^2+4^2}+2\right)^2=49$ $$C:58$$ D: If the two circles are externally tangent (min r), the sum of the radii is equal to the distance between the centers $$\sqrt{r}+2=\sqrt{41}$$ If the two circles are internally tangent (max r), the difference of the radii is equal to the distance between the centers $$\sqrt{r}-2=\sqrt{41}$$ $$\left(\sqrt{41}+2\right)^2+\left(\sqrt{41}-2\right)^2=90$$ Final: $$14+7+58+90=169$$
10	$30\sqrt{3}$	A: $$\cos^4(x)=\left(\frac{e^{ix}+e^{-ix}}{2}\right)^4=\frac{e^{4ix}+4e^{2ix}+6+4e^{-2ix}+e^{-4ix}}{16}$$ $$=\frac{\cos(4x)}{8}+\frac{\cos(2x)}{2}+\frac{3}{8}$$ $$A=\frac{1}{64}+\frac{1}{4}+\frac{9}{64}=\frac{26}{64}=\frac{13}{32}\rightarrow 45$$ B: Note that $$\int_0^{\frac{\pi}{2}}\cos^{2n}(x)\,dx=a_{2n,0}\cdot\frac{\pi}{2}$$ Also, for $\cos^{2n+1}(x)$, there is no constant term. Thus, the summation is

		$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1+\frac{\cos^2(x)}{4}+\frac{\cos^4(x)}{16}\ldots dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{1}{1-\frac{\cos^2(x)}{4}}dx$$ Multiplying the top and bottom by $\sec^2(x)$ gives $$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sec^2(x)}{\sec^2(x)-\frac{1}{4}}dx=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sec^2(x)}{\tan^2(x)+\frac{3}{4}}dx$$ Substituting $u=\tan(x)$, $$\frac{2}{\pi}\int_0^{\infty}\frac{1}{u^2+\frac{3}{4}}du=\frac{2}{\pi}\cdot\frac{2}{\sqrt{3}}\cdot\frac{\pi}{2}=\frac{2}{\sqrt{3}}$$ Final: $$45\cdot\frac{2\sqrt{3}}{3}=30\sqrt{3}$$
11	16	A: $$u=x^3+1, du=3x^2\,dx$$ $$\frac{1}{3}\int_1^2\frac{du}{u}=\frac{\ln 2}{3}$$ $$a_1+a_2+a_3=1+3+2=6$$ B: $$u=1-x^2, du=-2xdx$$ $$\int_0^1 x\sqrt{1-x^2}\,dx=\frac{1}{2}\int_0^1\sqrt{u}\,du=\frac{1}{3}u\sqrt{u}\Big]_0^1=\frac{1}{3}$$ $$b_1+b_2=1+3=4$$ C: $$\int_1^e 4x\ln x\,dx=2x^2\ln x]_1^e-\int_1^e 2x\,dx=2e^2-(e^2-1)=e^2+1$$ $$c_1+c_2+c_3=1+0+1=2$$ D: $$\frac{3x-4}{x^2-x-6}=\frac{M}{x-3}+\frac{N}{x+2}\to M(x+2)+N(x-3)=3x-4$$ $$x=-2:\ -5N=-10\to N=2$$ $$x=3: 5M=5\to M=1$$ $$\int_{-1}^2\left(\frac{1}{x-3}+\frac{2}{x+2}\right)dx=\ln\|(x-3)(x+2)^2\|]_{-1}^2=\ln 16-\ln 4=2\ln 2$$ $$d_1+d_2=2+2=4$$ Final: $$6+4+2+4=16$$
12	-36	A: When $k=0$, this is a degenerate hyperbola (two intersecting lines). B: When $x=y=0$, $k=-6$. Using implicit differentiation,

		$$\left(2+3\frac{dy}{dx}\right)(3x+2y+6)+(2x+3y-1)\left(3+2\frac{dy}{dx}\right)=0$$ $$x=y=0\to\left(2+3\frac{dy}{dx}\right)\cdot 6-\left(3+2\frac{dy}{dx}\right)=0$$ $$12+18\frac{dy}{dx}-3-2\frac{dy}{dx}=0$$ $$\frac{dy}{dx}=-\frac{9}{16}$$ C: The other leg of a right triangle with hypotenuse 20 and one leg 16 is 12. D: Differentiating the Pythagorean Theorem, $aa'+bb'=cc'$. The ladder is not changing length, so $aa'=-bb'$. $$a=12, b=16, b'=-4\to 12a'=64\to a'=\frac{16}{3}$$ Final: $$0+\left(-\frac{9}{16}\right)\cdot 12\cdot\frac{16}{3}=-36$$
13	$5-\ln\left(\frac{5}{3}\right)$	Sum = $$\int_1^2\frac{4x^4+3x^3+2x^2+x}{x^3+x^2+x+1}=4x-1-\frac{2x}{x^2+1}+\frac{1}{x+1}$$ $$=2x^2-x-\ln(x^2+1)+\ln(x+1)$$ $$=5-\ln(5)+\ln(2)+\ln(3)-\ln(2)=5-\ln\left(\frac{5}{3}\right)$$
14	$\frac{19}{4}+\sqrt{2}$	A: Writing the sequences recursively, $$a_0=1, b_0=1, a_n=a_{n-1}+2b_{n-1}, b_n=a_{n-1}+b_{n-1}$$ Let $\lim_{n\to\infty}\frac{a_n}{b_n}=S$ $$\frac{a_n}{b_n}=\frac{a_{n-1}+2b_{n-1}}{a_{n-1}+b_{n-1}}=\frac{\left(\frac{a_{n-1}}{b_{n-1}}\right)+2}{\left(\frac{a_{n-1}}{b_{n-1}}\right)+1}$$ $$S=\frac{S+2}{S+1}\to S=\sqrt{2}$$ B: $$a_n=\frac{\left(1+\sqrt{2}\right)^n+\left(1-\sqrt{2}\right)^n}{2}$$ Substituting this in, $$\frac{a_n}{3^n}=\frac{1}{2}\cdot\left(\frac{1+\sqrt{2}}{3}\right)^n+\frac{1}{2}\cdot\left(\frac{1-\sqrt{2}}{3}\right)^n$$ Sum = $$\left(\frac{1}{1-\frac{1+\sqrt{2}}{3}}+\frac{1}{1-\frac{1-\sqrt{2}}{3}}\right)\cdot\frac{1}{2}=3$$

$$3 - a_0 = 2$$

C:

Let $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = S$

$$\frac{c_{n-1} + 2c_{n-2} + 2}{c_{n-1}} = \frac{c_n}{c_{n-1}}$$

$$1 + \frac{2}{S} = S \to S = 2$$

D:

Let the sum be S

$$2S = \frac{2c_1}{3} + \frac{2c_2}{9} \ldots$$

$$3S = c_1 + \frac{c_2}{3} + \frac{c_3}{9} \ldots$$

$$1 = \frac{2}{3} + \frac{2}{9} \ldots$$

$$5S + 1 = c_1 + \frac{2c_1 + c_2 + 2}{3} + \frac{2c_2 + c_3 + 2}{9} \ldots$$

$$5S + 1 = \frac{c_3}{3} + \frac{c_4}{9} \ldots = 9S - 2$$

$$\to S = \frac{3}{4}$$

Final:

$$\sqrt{2} + 2 + 2 + \frac{3}{4} = \frac{19}{4} + \sqrt{2}$$