FAMAT Geometry Study Guide

By: Kaitlyn Lee

Edited by Saber Lian

Contents

1 Points, Lines, Planes and Angles

1.1 Basic Definitions

Definition 1.1. *A **point** has no size, only location. Denoted by capital letters (e.g., A).*

Definition 1.2. *A **line** extends infinitely in both directions. Named by two points (e.g., $\overleftrightarrow{AB}$).*

Definition 1.3. *A **plane** is a flat surface extending infinitely in all directions.*

1.2 Segments, Rays, Midpoint, Distance

Definition 1.4. *A **segment** is part of a line between two points ($\overline{AB}$).*
***Midpoint** of (x_1, y_1) and (x_2, y_2): $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$*
***Distance** between points: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$*

Definition 1.5. *A **ray** starts at a point and extends infinitely ($\overrightarrow{AB}$).*

1.3 Angles

Definition 1.6. *An **angle** is formed by two rays sharing an endpoint. Measured in degrees (°).*
Right angle $= 90°$, Straight angle $= 180°$

1.4 Postulates and Theorems

Postulate 1.1 (Two Points)**.** *Through any two points, there is exactly one line.*

Postulate 1.2 (Three Noncollinear Points)**.** *Through any three noncollinear points, there is exactly one plane.*

Postulate 1.3 (Plane Intersection)**.** *If two planes intersect, then their intersection is a line.*

Counting Lines and Planes from Points

- **Number of Unique Lines from n Points**:

$$\text{Lines} = \binom{n}{2} = \frac{n(n-1)}{2}$$

This counts all distinct pairs of points, since a line is determined by two points. Assumes no three points are collinear.

- **Number of Unique Planes from n Points**:

$$\text{Planes} = \binom{n}{3} = \frac{n(n-1)(n-2)}{6}$$

Each plane is determined by a unique triplet of noncollinear points. Assumes no four points are coplanar and no three are collinear.

- **Note on Degeneracy**: If some points are collinear or coplanar, the actual number of distinct lines or planes may be less than the calculated maximum.

Postulate 1.4 (Three Points)**.** *Through any three non-collinear points there is exactly one plane.*

Theorem 1.1 (Maximum Number of Regions by Lines)**.** *The maximum number of regions into which n lines can divide a plane is*

$$R = \frac{n^2+n+2}{2}$$

Theorem 1.2 (Maximum Number of Regions by Planes in 3D Space)**.** *The maximum number of regions into which n planes can divide space is*

$$R = \frac{n^3+5n+6}{6}$$

1.5 Key Terms

Definition 1.7. ***Collinear*** *points lie on the same line.*
Coplanar *points lie on the same plane.*
Intersection *is where geometric figures meet.*

2 Deductive Reasoning

2.1 Conditional Statements

A **conditional statement** has the form: *If p, then q*, written as $p \to q$.

- **Hypothesis (p)**: the "if" part
- **Conclusion (q)**: the "then" part

Related Statements:

- **Converse**: $q \rightarrow p$
- **Inverse**: $\neg p \rightarrow \neg q$
- **Contrapositive**: $\neg q \rightarrow \neg p$

Logical Relationships:

- A conditional and its contrapositive are **logically equivalent**.
- The converse and the inverse are **logically equivalent**.

Example:

- p: It is raining.
- q: The ground is wet.
- Conditional: If it is raining, then the ground is wet. $(p \rightarrow q)$
- Converse: If the ground is wet, then it is raining. $(q \rightarrow p)$
- Inverse: If it is not raining, then the ground is not wet. $(\neg p \rightarrow \neg q)$
- Contrapositive: If the ground is not wet, then it is not raining. $(\neg q \rightarrow \neg p)$

2.2 Logical Operators

- **Negation** $(\neg p)$: The opposite of a statement.
- **Conjunction** $(p \wedge q)$: True only if both p and q are true.
- **Disjunction** $(p \vee q)$: True if at least one of p or q is true.
- **Conditional** $(p \rightarrow q)$: False only if p is true and q is false.
- **Biconditional** $(p \leftrightarrow q)$: True if p and q are both true or both false.

2.3 Truth Tables

Negation

p	$\neg p$
T	F
F	T

Conjunction ($p \wedge q$)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction ($p \vee q$)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional ($p \to q$)

p	q	$p \to q$
T	T	T
T	F	**F**
F	T	T
F	F	T

Biconditional ($p \leftrightarrow q$)

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

2.4 De Morgan's Laws

- $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

2.5 Laws of Logic (Examples of Deductive Reasoning)

- **Law of Detachment**: If $p \to q$ is true and p is true, then q is true.
- **Law of Contrapositive**: If $p \to q$ is true and $\neg q$ is true, then $\neg p$ is true.
- **Law of Syllogism (Chain Rule)**: If $p \to q$ and $q \to r$, then $p \to r$.

3 Parallel Lines and Planes

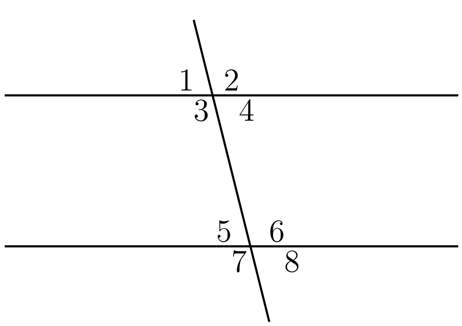
3.1 Definitions

Definition 3.1. ***Parallel lines*** *never intersect.* ***Skew lines*** *are non-parallel and non-intersecting in 3D.*

3.2 Properties of Parallel Lines

Theorem 3.1. *If parallel lines are cut by a transversal:*

- *Corresponding angles equal*
- *Alternate interior angles equal*
- *Consecutive interior angles supplementary*



Angle Relationships:
Corresponding: $\angle 1 \cong \angle 5$, $\angle 2 \cong \angle 6$
Alternate Interior: $\angle 3 \cong \angle 6$, $\angle 4 \cong \angle 5$
Same-Side Interior: $\angle 3 + \angle 6 = 180°$,
$\angle 4 + \angle 5 = 180°$

3.3 Proving Lines Parallel

Theorem 3.2. *If any of these angle pairs are congruent, lines are parallel:*

- *Corresponding angles*
- *Alternate interior angles*
- *Alternate exterior angles*

3.4 Angles in Triangles

Theorem 3.3. *Sum of angles in a triangle is* $180°$

3.5 Angles in Polygons

Theorem 3.4. *For n-sided polygon:*

- *Sum interior angles:* $180(n-2)°$
- *Sum exterior angles:* $360°$
- *Number of diagonals:* $\frac{n(n-3)}{2}$
- *Each interior angle (regular):* $\frac{(n-2)\times 180°}{n}$

Sides	Name
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon
11	Undecagon
12	Dodecagon
n	n-gon

4 Congruent Triangles

4.1 Triangle Classification

- **Scalene**: All sides different lengths
- **Isosceles**: Two equal sides, two equal base angles
- **Equilateral**: All sides equal, all angles $60°$
- **Acute**: All angles $< 90°$
- **Right**: One $90°$ angle
- **Obtuse**: One angle $> 90°$

4.2 Using Congruence Theorems and CPCTC

Congruence Theorems provide criteria to prove that two triangles are congruent by comparing specific parts (sides and angles). To use these theorems:

- **SSS (Side-Side-Side):** If all three pairs of corresponding sides of two triangles are congruent, then the triangles are congruent.
- **SAS (Side-Angle-Side):** If two pairs of corresponding sides and the included angle (the angle between those two sides) are congruent, then the triangles are congruent.
- **ASA (Angle-Side-Angle):** If two pairs of corresponding angles and the included side (the side between those angles) are congruent, then the triangles are congruent.
- **AAS (Angle-Angle-Side):** If two pairs of corresponding angles and a non-included side are congruent, then the triangles are congruent.
- **HL (Hypotenuse-Leg):** For right triangles only — if the hypotenuse and one leg of one right triangle are congruent to the corresponding parts of another right triangle, then the triangles are congruent.

How to apply these in proofs:

1. Identify the triangles you want to prove congruent.
2. Determine which sides or angles you know to be congruent from given information or previously proven facts.

3. Match those known congruent parts to one of the congruence theorems above.
4. State the appropriate theorem (SSS, SAS, ASA, AAS, or HL) and conclude the triangles are congruent.

CPCTC (Corresponding Parts of Congruent Triangles are Congruent) is used after you have proven two triangles congruent. It allows you to conclude that any corresponding sides or angles of those triangles are also congruent.

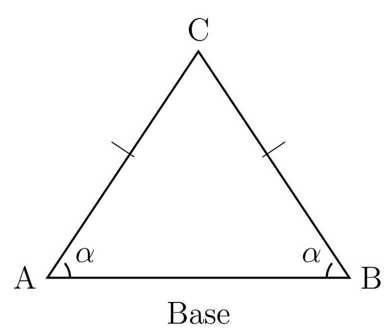
In proofs:

- Once you prove $\triangle ABC \cong \triangle DEF$, you can say $\angle A \cong \angle D$, $AB \cong DE$, etc., for any corresponding parts.
- This is useful to prove additional equalities or relationships needed to solve problems or complete the proof.

4.3 Isosceles Triangle Theorems

Theorem 4.1 (Base Angles Theorem)**.** *In an isosceles triangle, the base angles are congruent.*

Theorem 4.2 (Converse of Base Angles Theorem)**.** *If two angles of a triangle are congruent, then the sides opposite those angles are congruent.*



4.4 Special Segments

- **Median:** A segment from a vertex to the midpoint of the opposite side.
- **Altitude:** A perpendicular segment from a vertex to the line containing the opposite side.

- **Perpendicular bisector:** A line that is perpendicular to a side and passes through its midpoint.
- **Angle bisector:** A segment or ray that divides an angle into two congruent angles.

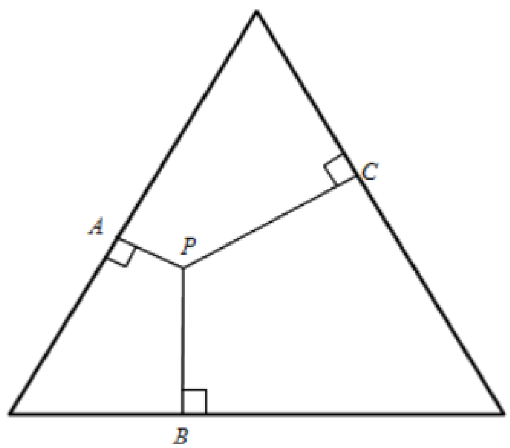
  Theorem 4.3 (Angle Bisector Theorem)**.** *If AD bisects $\angle A$, then*

$$\frac{AB}{BD} = \frac{AC}{CD}$$

- **Viviani's Theorem:** If P is any point inside an equilateral triangle ABC, then the sum of the perpendicular distances from P to the three sides is equal to the altitude h of the triangle:

$$PQ + PR + PS = h$$

  where PQ, PR, PS are the perpendicular distances from P to the sides of the triangle.

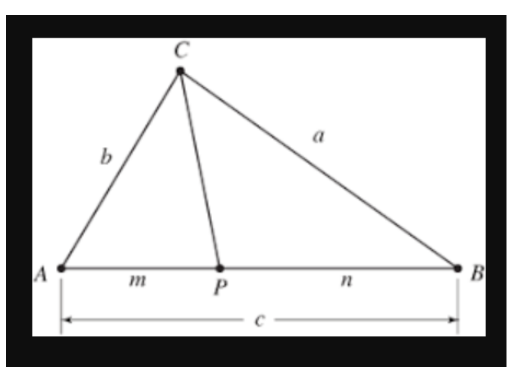


- **Stewart's Theorem:** For any cevian AD of triangle ABC, where D lies on BC, let:

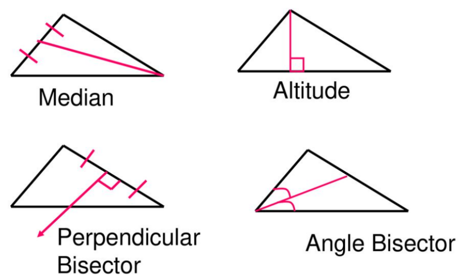
$$AB = c, \quad AC = b, \quad BC = a, \quad BD = m, \quad DC = n, \quad AD = d$$

  Then Stewart's theorem states:

$$b^2m + c^2n = a(d^2 + mn)$$



How to use these: - Use *medians* to find the centroid, which balances the triangle. - Use *altitudes* to find the orthocenter, important for height and area problems. - Use *perpendicular bisectors* to find the circumcenter, the center of the circumscribed circle. - Use *angle bisectors* to find the incenter, the center of the inscribed circle (incircle) of the triangle.



4.5 Exterior Angle Theorem

Theorem 4.4. *The measure of an exterior angle of a triangle is equal to the sum of the measures of the two non-adjacent interior angles.*

How to use: Use this theorem to find unknown angle measures outside the triangle by relating them to the interior angles. It is especially useful in proofs involving angle sums or inequalities.

4.6 Triangle Centers

Center	**Definition**
Centroid	Intersection of medians
Orthocenter	Intersection of altitudes
Circumcenter	Intersection of perpendicular bisectors
Incenter	Intersection of angle bisectors

How to use: These points have special properties and appear in many geometric constructions and proofs. Their locations vary based on the triangle type:

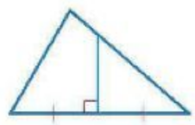
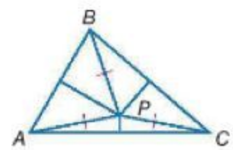
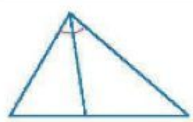
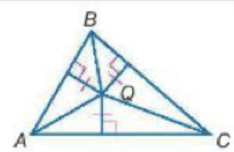
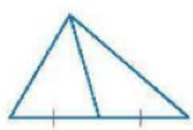
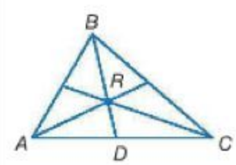
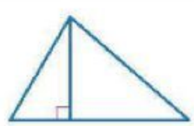
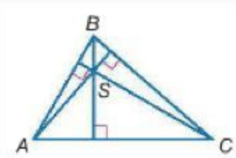
- **Centroid:** Always located *inside* the triangle. It divides each median into a 2:1 ratio, with the longer segment between the vertex and the centroid.
- **Orthocenter:**
  - *Acute triangle:* Inside the triangle.

- *Right triangle:* At the vertex of the right angle.
- *Obtuse triangle:* Outside the triangle.

- **Circumcenter:**
  - *Acute triangle:* Inside the triangle.
  - *Right triangle:* At the midpoint of the hypotenuse.
  - *Obtuse triangle:* Outside the triangle.

  It is equidistant from all vertices and is the center of the circumscribed circle.

- **Incenter:** Always located *inside* the triangle. It is equidistant from all sides and is the center of the inscribed circle (incircle).

Name	Example	Point of Concurrency	Special Property	Example
perpendicular bisector		circumcenter	The circumcenter P of $\triangle ABC$ is equidistant from each vertex.	
angle bisector		incenter	The incenter Q of $\triangle ABC$ is equidistant from each side of the triangle.	
median		centroid	The centroid R of $\triangle ABC$ is two thirds of the distance from each vertex to the midpoint of the opposite side.	
altitude		orthocenter	The lines containing the altitudes of $\triangle ABC$ are concurrent at the orthocenter S.	

5 Quadrilaterals

5.1 Parallelograms

Theorem 5.1. *Properties:*

- *Opposite sides parallel and equal*

- *Opposite angles equal*
- *Consecutive angles supplementary*
- *Diagonals bisect each other*

5.2 Proving Parallelograms

Theorem 5.2. *A quadrilateral is a parallelogram if:*

- *Congruent opposite sides*
- *Congruent opposite angles*
- *Supplementary consecutive angles*
- *If the quadrilateral has one right angle, then it has four right angles*
- *Bisecting diagonals*
- *Each diagonal separates the parallelogram into two congruent triangles*

5.3 Special Parallelograms

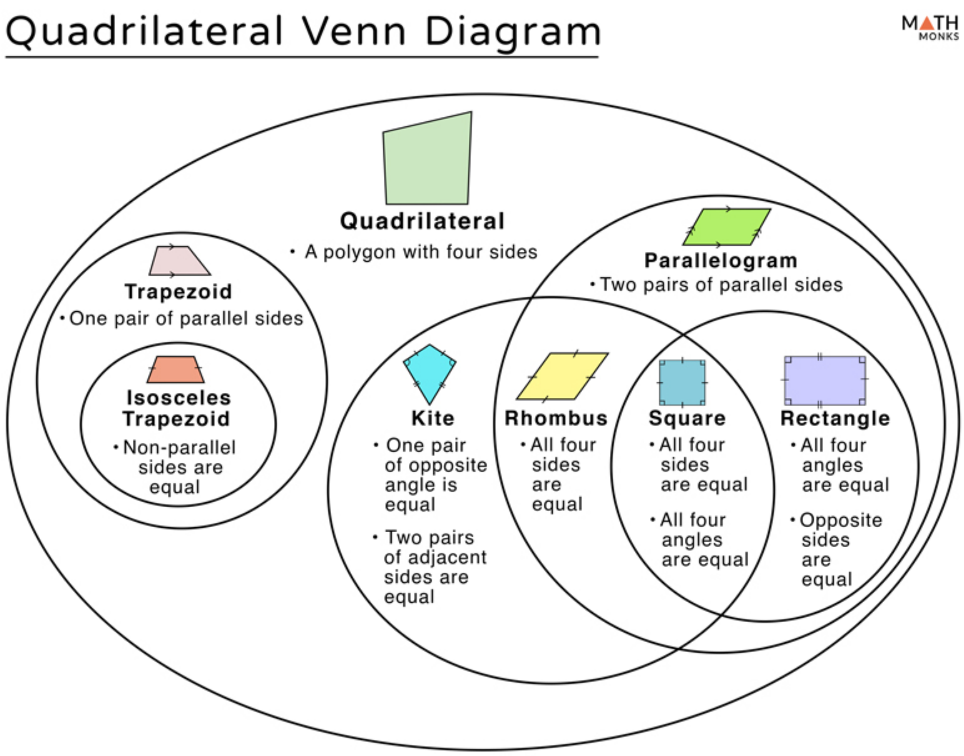
- **Rectangle**: All angles 90°, diagonals equal
- **Rhombus**: All sides equal, diagonals perpendicular
- **Square**: Both rectangle and rhombus

5.4 Trapezoids

Definition 5.1. *Exactly one pair parallel sides (bases)*

Theorem 5.3. *Isosceles trapezoid:*

- *Legs equal*
- *Base angles equal*
- *Diagonals equal*
- *Median length:* $\frac{b_1+b_2}{2}$
- *Segment joining diagonal midpoints:* $\frac{|b_1-b_2|}{2}$



5.5 Kites

Theorem 5.4. *Properties:*

- *Two pairs adjacent equal sides*
- *Diagonals perpendicular*
- *One diagonal bisected*
- *One pair opposite angles equal*

5.6 Rectangle

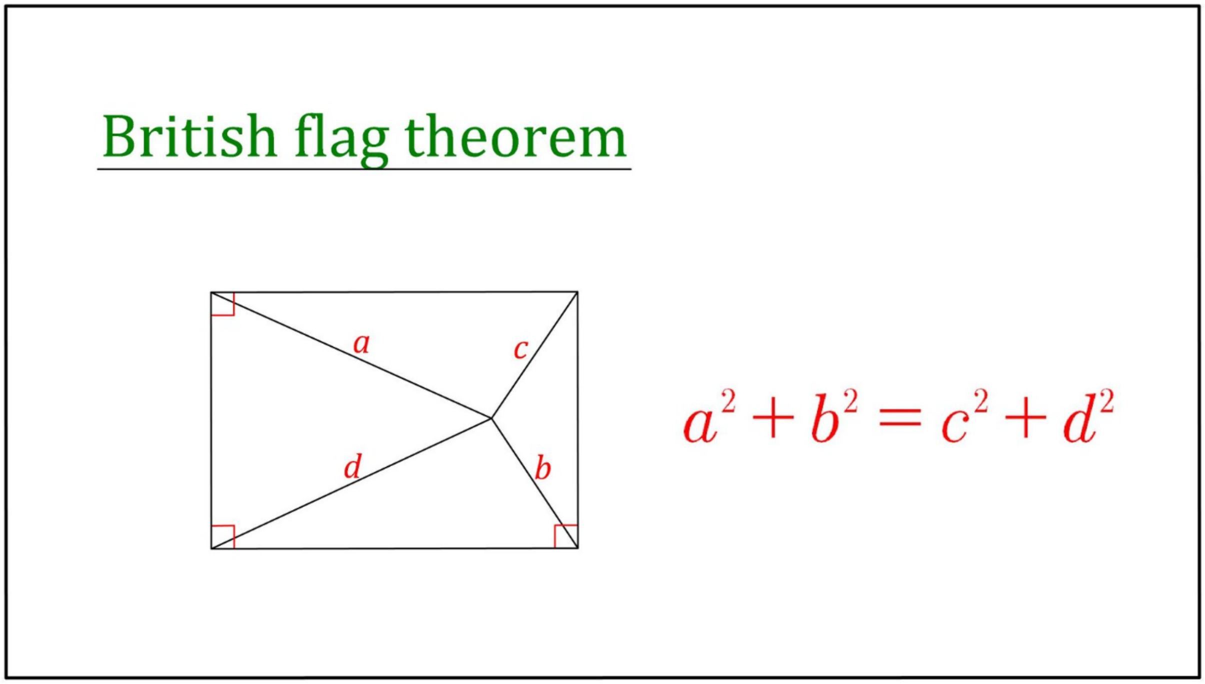
A rectangle is a parallelogram with all angles 90° and equal diagonals.

British Flag Theorem

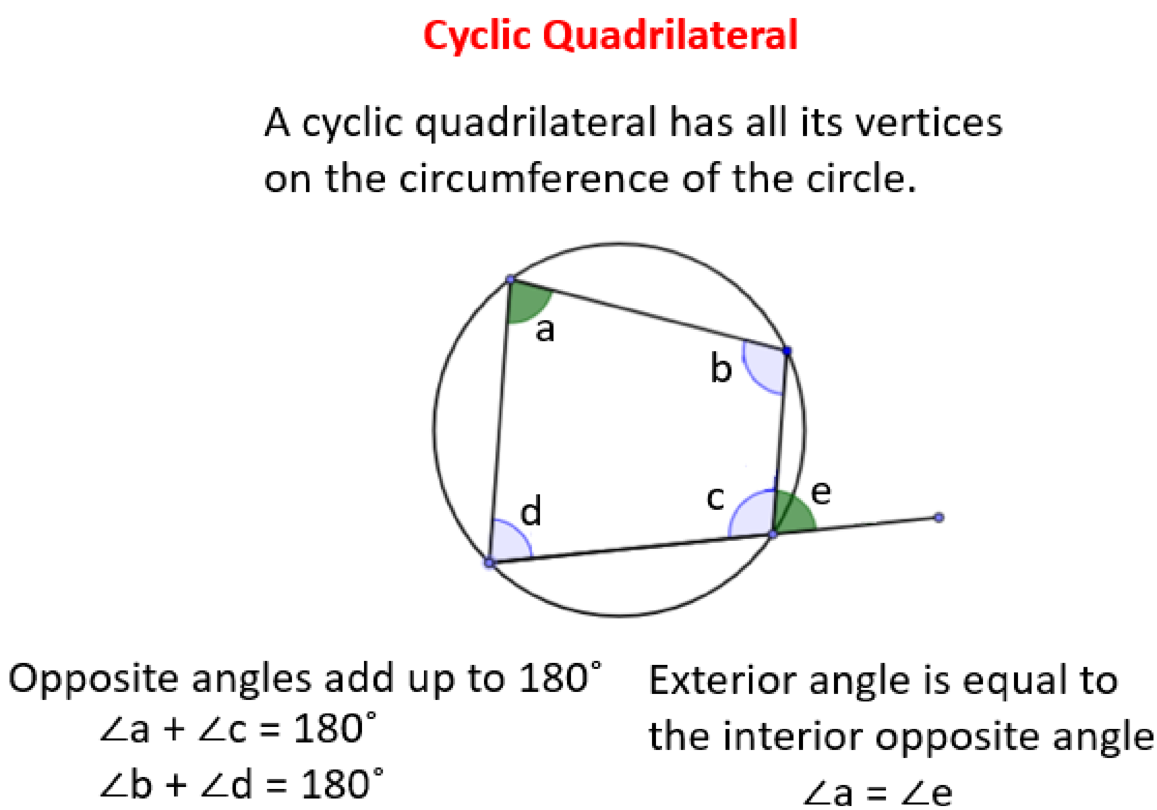
For any point P inside rectangle $ABCD$, the sum of the squares of the distances from P to two opposite corners equals the sum of the squares of the distances to the other two opposite corners:

$$PA^2 + PC^2 = PB^2 + PD^2$$

where PA, PB, PC, PD are the distances from P to vertices A, B, C, D respectively.



5.7 Cyclic Quadrilaterals



Definition: A cyclic quadrilateral is a quadrilateral whose vertices all lie on a single circle (called the *circumcircle*). In other words, it is *inscribed* in a circle.

How to identify:

- If a quadrilateral has opposite angles that are supplementary (i.e., they add up to 180°), then it is cyclic.
- If all vertices lie on a circle, then it is cyclic.
- If it satisfies Ptolemy's Theorem, it must be cyclic.

Theorem 5.5 (Opposite Angles)**.** *In a cyclic quadrilateral, the opposite angles are supplementary:*

$$\angle A + \angle C = 180^\circ, \quad \angle B + \angle D = 180^\circ$$

Theorem 5.6 (Brahmagupta's Formula)**.** *The area of a cyclic quadrilateral with side lengths a, b, c, and d is given by:*

$$Area = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

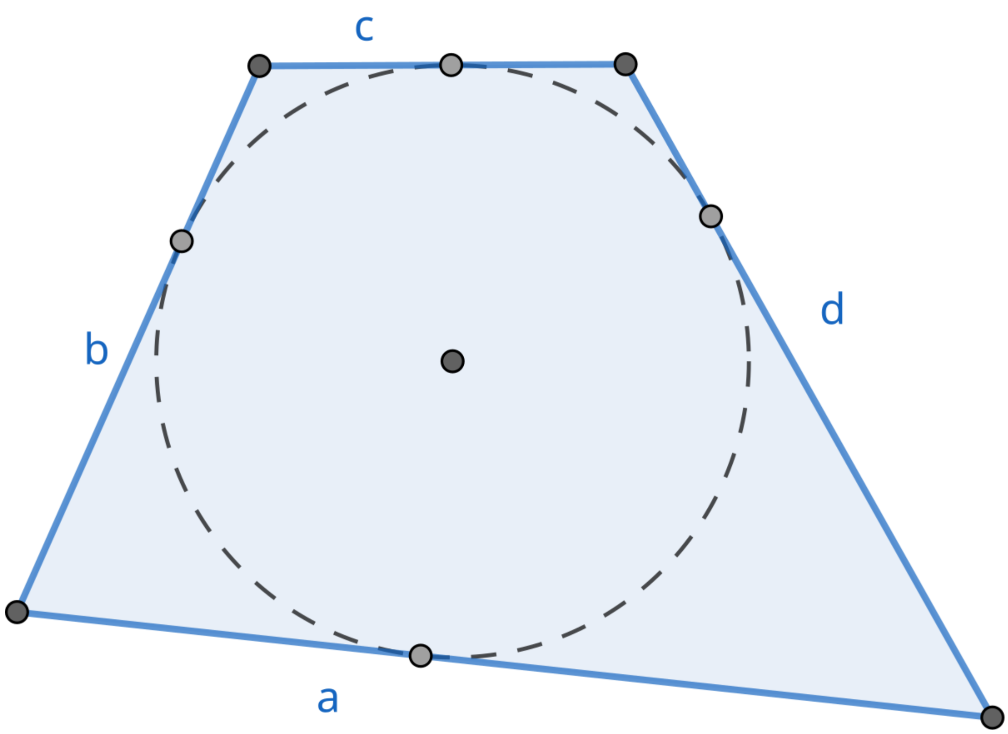
where $s = \frac{a+b+c+d}{2}$ is the semiperimeter.

Theorem 5.7 (Ptolemy's Theorem)**.** *In a cyclic quadrilateral with vertices A, B, C, and D in order:*

$$AC \cdot BD = AB \cdot CD + AD \cdot BC$$

This relates the diagonals to the products of the opposite sides.

5.8 Tangential Quadrilaterals



Definition: A tangential quadrilateral is a quadrilateral that has an *incircle* — a circle that is tangent to all four of its sides.

How to identify:

- If a quadrilateral has all four sides tangent to a single circle, it is tangential.
- If the sums of the lengths of opposite sides are equal, the quadrilateral is tangential:

$$a + c = b + d$$

Theorem 5.8 (Side Sum Condition)**.** *For a tangential quadrilateral with sides a, b, c, and d:*

$$a + c = b + d$$

Theorem 5.9 (Area Formula). *The area of a tangential quadrilateral is:*

$$Area = r \cdot s$$

where r is the inradius and s is the semiperimeter: $s = \frac{a+b+c+d}{2}$.

6 Inequalities in Geometry

6.1 Indirect Proofs (Proof by Contradiction)

To prove a statement is true, assume the *opposite* is true and then show that this leads to a contradiction. This technique is especially useful for inequality proofs in triangles.

Example: Prove that in $\triangle ABC$, $\angle C$ is the largest if AB is the longest side.

- Assume the opposite: $\angle C$ is not the largest angle.
- Then either $\angle A$ or $\angle B$ is larger.
- But in any triangle, the larger angle is opposite the longer side.
- That would mean either BC or AC is longer than AB, contradicting our assumption that AB is the longest.
- So, $\angle C$ must be the largest.

6.2 Triangle Inequality Theorem (One-Triangle Inequality)

Theorem 6.1 (Triangle Inequality). *In any triangle, the sum of the lengths of any two sides is greater than the third side.*

$$AB + BC > AC, \quad AB + AC > BC, \quad BC + AC > AB$$

This must hold for all triangles. If it doesn't, the figure cannot form a triangle.

Note: This is known as a *one-triangle inequality* because all the sides being compared belong to a single triangle.

- Also remember: In any triangle, the **longest side** is always opposite the **largest angle**, and vice versa.

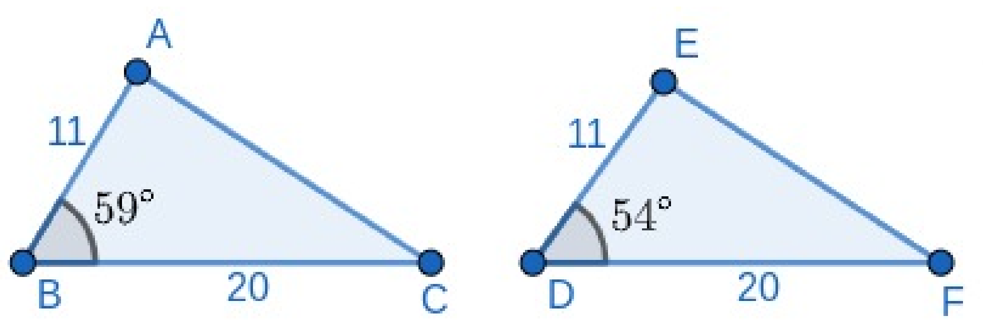
6.3 Angle–Side Relationships in Triangles

These help determine the type of triangle based on the side lengths:

- **Acute triangle:** $a^2 + b^2 > c^2$ (All angles are less than 90°)
- **Right triangle:** $a^2 + b^2 = c^2$ (One angle is 90°)
- **Obtuse triangle:** $a^2 + b^2 < c^2$ (One angle is greater than 90°)

Assumes c is the longest side.

6.4 Hinge Theorem (SAS Inequality Theorem)



Theorem 6.2 (Hinge Theorem)**.** *If two triangles share two sides of equal length, but one triangle has a larger included angle, then the side opposite that angle is longer.*

If: $AB = DE$, $AC = DF$, *and* $\angle A > \angle D$ ***Then:*** $BC > EF$

Explanation: This theorem is like a "geometric hinge." If you imagine opening a pair of scissors — with the blades being the fixed sides — the more you open the angle, the farther apart the tips get.

6.5 Two-Triangle Inequality (Converse of Hinge Theorem)

Theorem 6.3 (Converse of the Hinge Theorem)**.** *If two triangles share two sides of equal length, but one has a longer third side, then the included angle of that triangle is greater.*

If: $AB = DE$, $AC = DF$, *and* $BC > EF$ ***Then:*** $\angle A > \angle D$

This is used in reverse situations — when comparing angles instead of sides.

7 Similar Polygons

7.1 Ratios and Proportions

A **ratio** is a comparison of two quantities: $\frac{a}{b}$ or $a : b$. A **proportion** is an equation that equates two ratios:

$$\frac{a}{b} = \frac{c}{d} \quad \Leftrightarrow \quad ad = bc$$

7.2 Properties of Proportions

If $\frac{a}{b} = \frac{c}{d}$, then the following are also true:

- **Reciprocal Property:** $\frac{b}{a} = \frac{d}{c}$
- **Inversion (Means and Extremes):** $\frac{a}{c} = \frac{b}{d}$
- **Addition Property:** $\frac{a+b}{b} = \frac{c+d}{d}$
- **Subtraction Property:** $\frac{a-b}{b} = \frac{c-d}{d}$

7.3 Definition: Similar Polygons

Two polygons are **similar** if:

- All corresponding angles are congruent.
- All corresponding side lengths are in proportion.

$$\text{If } \triangle ABC \sim \triangle DEF, \text{ then: } \angle A \cong \angle D, \quad \angle B \cong \angle E, \quad \angle C \cong \angle F$$
$$\text{And } \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

7.4 Proving Triangles Similar

Triangle Similarity Postulates:

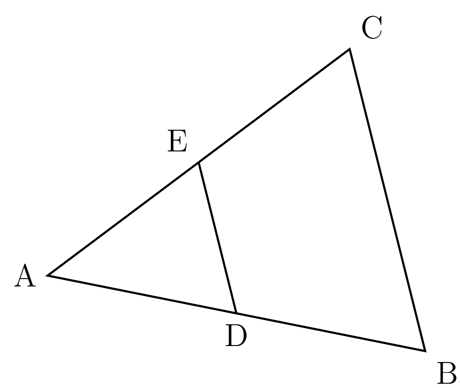
- **AA (Angle-Angle):** Two corresponding angles are equal.
- **SAS (Side-Angle-Side):** Two sides in proportion and the included angle is equal.
- **SSS (Side-Side-Side):** All three pairs of corresponding sides are proportional.

Example (AA): If $\angle A = \angle D$ and $\angle B = \angle E$, then $\triangle ABC \sim \triangle DEF$.

7.5 Proportional Segments Theorem

Theorem 7.1 (Basic Proportionality Theorem (Thales' Theorem))**.** *If a line is drawn parallel to one side of a triangle and intersects the other two sides, it divides those two sides proportionally.*

$$\textit{If } DE \parallel BC \textit{ in } \triangle ABC, \textit{ then } \frac{AD}{DB} = \frac{AE}{EC}$$



7.6 Scale Factor and Its Properties

If two similar figures have a scale factor of k, then:

- **Ratio of sides:** k
- **Ratio of perimeters:** k
- **Ratio of areas:** k^2
- **Ratio of volumes (if 3D):** k^3

Example: If two similar triangles have a side ratio of $2:3$, then:

$$\text{Area ratio} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}, \quad \text{Volume ratio (for solids)} = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

Tip: Always match corresponding sides and angles before applying any ratio or similarity rule. Use notation like $\triangle ABC \sim \triangle DEF$ to show correct order of correspondence.

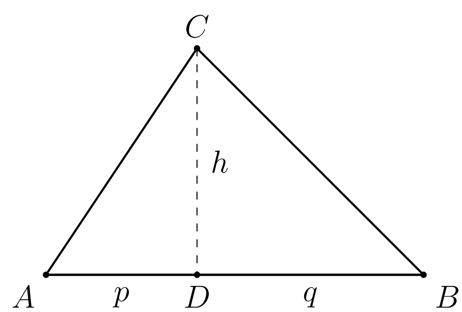
8 Right Triangles

8.1 Geometric Mean

Theorem 8.1. *In a right triangle with altitude drawn to the hypotenuse:*

- $h = \sqrt{pq}$ *(altitude is mean of segments on hypotenuse)*
- $a = \sqrt{pc}$ *(leg is mean of adjacent segment and whole hypotenuse)*
- $b = \sqrt{qc}$

where $c = p + q$*, and* h *is the altitude to the hypotenuse.*



8.2 Pythagorean Theorem

Theorem 8.2. *In a right triangle:* $a^2 + b^2 = c^2$

8.3 Converse: Classifying Triangles by Side Lengths

Given side lengths a, b, and c with c as the longest side:

- **Right Triangle:** $a^2 + b^2 = c^2$
- **Acute Triangle:** $a^2 + b^2 > c^2$
- **Obtuse Triangle:** $a^2 + b^2 < c^2$

8.4 Special Right Triangles

Type	Angles	Side Ratios
Isosceles Right Triangle	$45°, 45°, 90°$	$1 : 1 : \sqrt{2}$
Half an Equilateral Triangle	$30°, 60°, 90°$	$1 : \sqrt{3} : 2$

8.5 Pythagorean Triples

Common integer triples that satisfy $a^2 + b^2 = c^2$:

$$(3, 4, 5),\ (5, 12, 13),\ (7, 24, 25),\ (8, 15, 17),\ (9, 40, 41),\ (12, 35, 37)$$

Tip: Multiples of these are also valid (e.g., $(6, 8, 10)$ is a multiple of $(3, 4, 5)$).

8.6 Median to Hypotenuse

Theorem 8.3. *In a right triangle, the median to the hypotenuse equals half the hypotenuse.*

$$\textit{If } AB \textit{ is the hypotenuse, then } CM = \frac{1}{2}AB$$

8.7 Trigonometric Ratios

For a right triangle with angle θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \quad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

8.8 Applications

- Use trig ratios to solve for missing sides or angles
- Angle of elevation: angle above horizontal
- Angle of depression: angle below horizontal

Example (Elevation): A 20 ft ladder leans against a wall, forming a $65°$ angle with the ground. How high up the wall does it reach?

$$\sin(65°) = \frac{h}{20} \Rightarrow h = 20 \cdot \sin(65°) \approx 18.1 \text{ ft}$$

Example (Depression): From the top of a lighthouse 80 ft tall, the angle of depression to a boat in the water is 27°. How far is the boat from the base of the lighthouse?

$$\tan(27°) = \frac{80}{x} \Rightarrow x = \frac{80}{\tan(27°)} \approx 157.5 \text{ ft}$$

9 Circles (February/March)

9.1 Basic Formulas

- **Circumference**: $C = 2\pi r$ — the distance around the circle.
- **Area**: $A = \pi r^2$ — the space enclosed within the circle.

9.2 Types of Angles in Circles

Central Angles

- A central angle's measure is equal to its intercepted arc.
- $\angle AOB = \overset{\frown}{AB}$

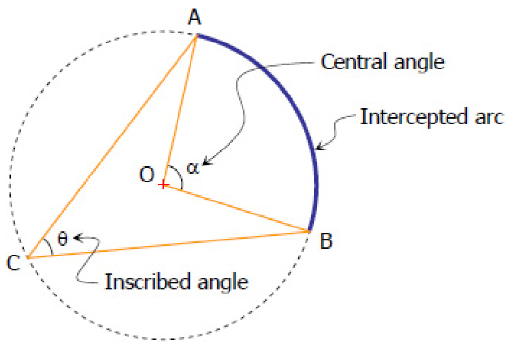
Inscribed Angles

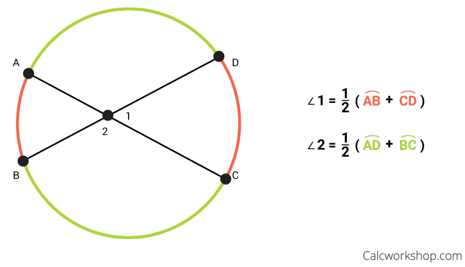
- An inscribed angle is half the measure of its intercepted arc.
- $\angle C = \frac{1}{2}\overset{\frown}{AB}$
- **Special Case:** If an inscribed angle intercepts a semicircle (i.e., a diameter), it's a right angle.

Angle Formed by Intersecting Chords

- The angle formed by two intersecting chords equals half the sum of the intercepted arcs.
- $\angle 1 = \frac{1}{2}(\overset{\frown}{AB} + \overset{\frown}{CD})$

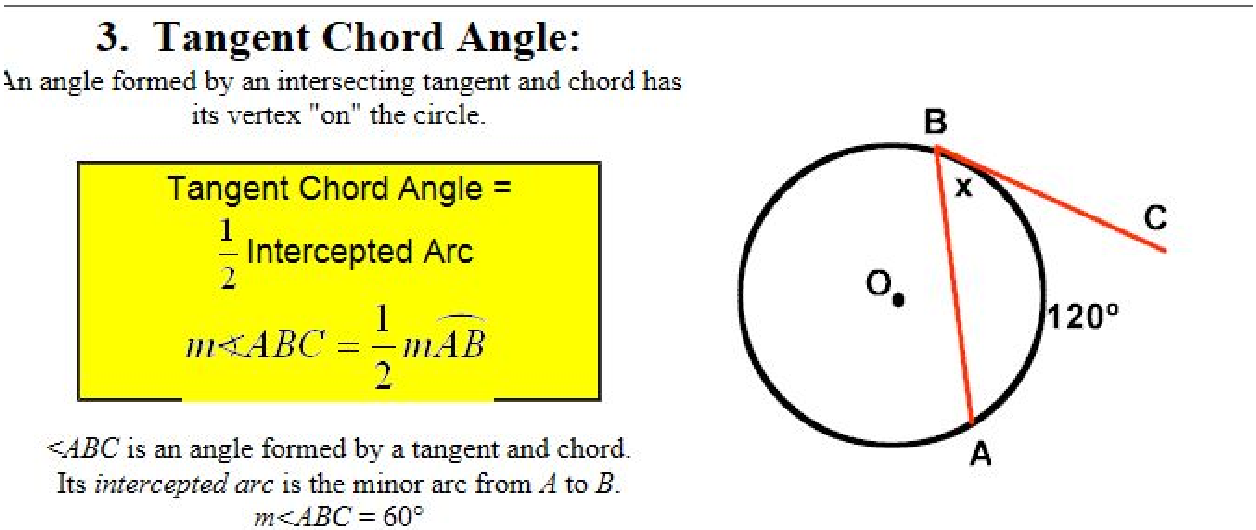
- $\angle 2 = \frac{1}{2}(\overset{\frown}{AD} + \overset{\frown}{CB})$





Angle Formed by a Chord and a Tangent

- When a tangent and a chord intersect at the point of tangency, the angle between them is **half the measure of the intercepted arc**.
- **Formula:** $\angle ABC = \frac{1}{2}\overset{\frown}{AC}$, where:
  - Point B is the point of tangency,
  - AB is the chord,
  - BC is the tangent,
  - $\overset{\frown}{AC}$ is the intercepted arc.



Angle Formed by Two Secants (Outside the Circle)

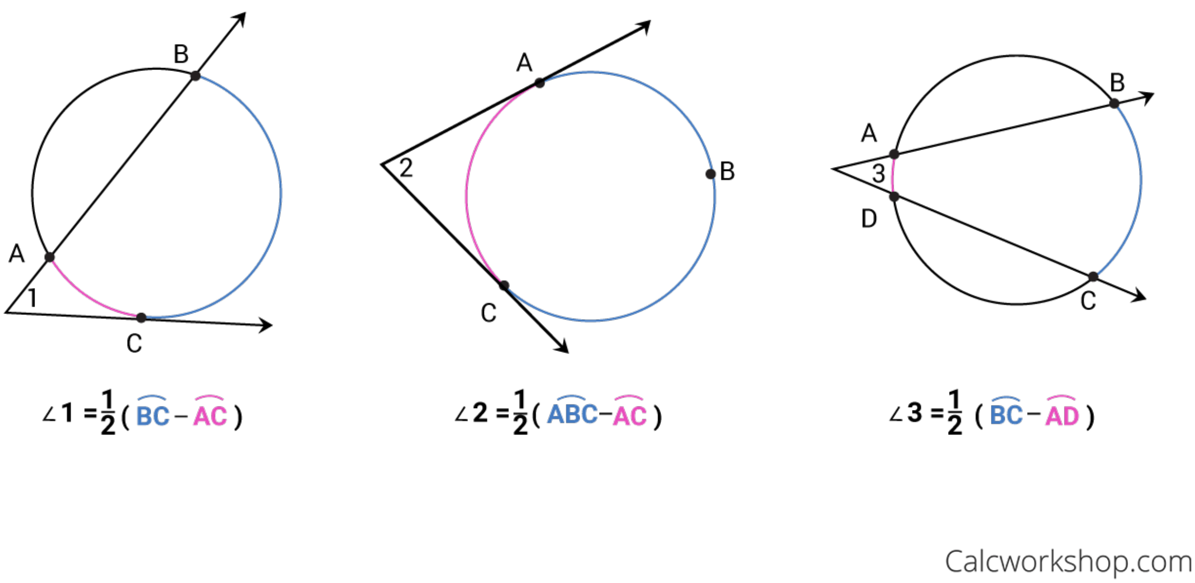
- The angle is half the difference of the intercepted arcs.
- $\angle M = \frac{1}{2}(\overset{\frown}{OQ} - \overset{\frown}{NP})$

Angle Formed by Two Tangents

- $\angle C = \frac{1}{2}(\overset{\frown}{ADB} - \overset{\frown}{AB})$
- The lengths from the external point to the points of tangency are equal: $CA = CB$

Angle Formed by a Tangent and a Secant

- $\angle A = \frac{1}{2}(\overset{\frown}{BD} - \overset{\frown}{CB})$



Angle Formed by Clock Hands

The angle between the hour and minute hands of a clock at h hours and m minutes is given by:

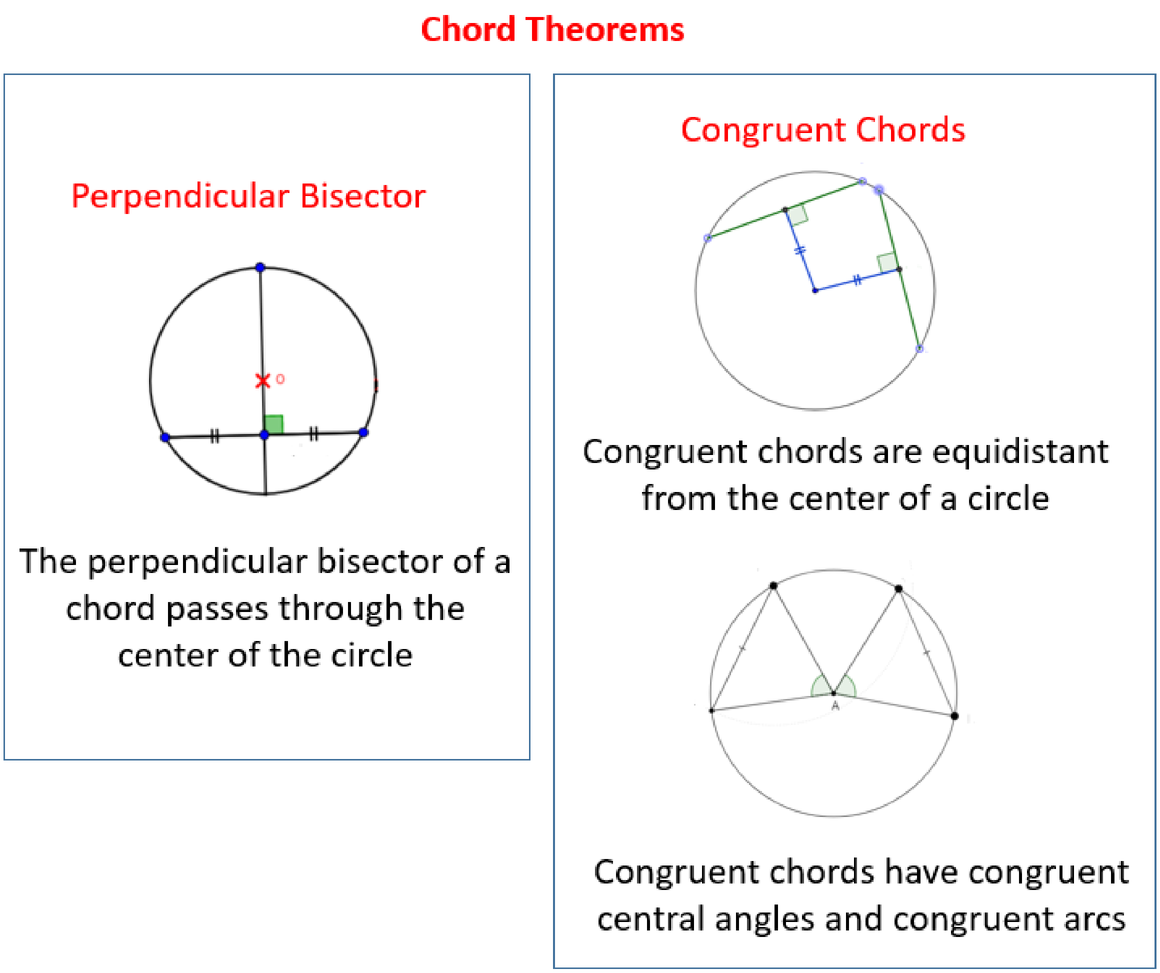
$$\theta = \left|30h - \frac{11}{2}m\right|$$

where

- h is the hour (in 12-hour format)
- m is the minutes past the hour

9.3 Chord Properties

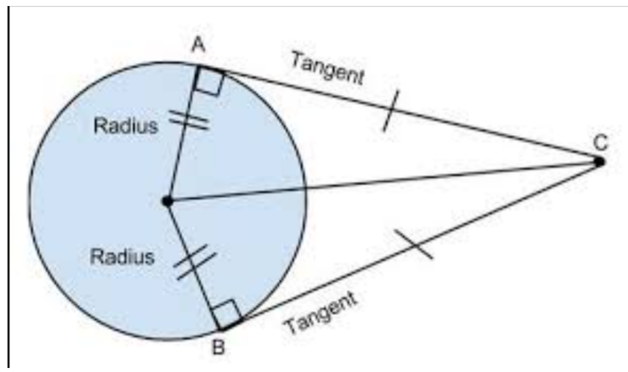
- A perpendicular from the center to a chord bisects the chord.
- Equal chords $\Leftrightarrow$ equal arcs.



9.4 Tangent Properties

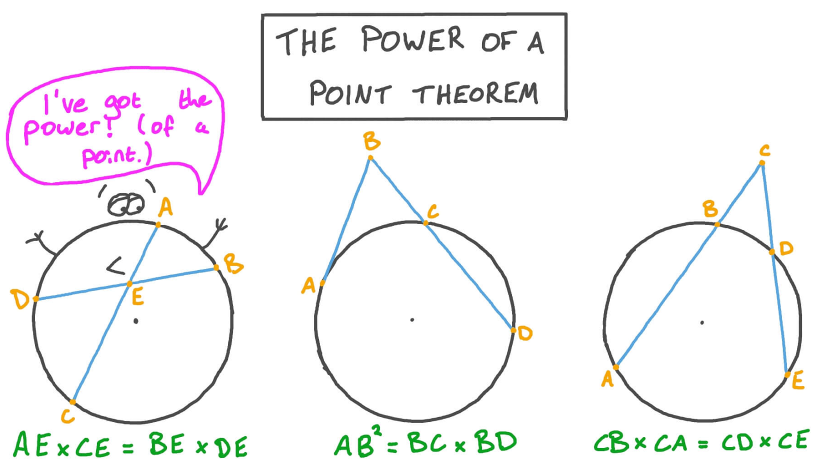
- A tangent is perpendicular to the radius at the point of tangency.

- Two tangents drawn from the same external point are equal in length.



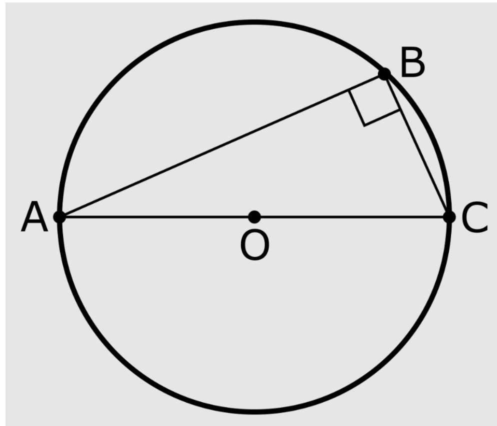
9.5 Power of a Point Theorems

- **Two Secants:** $PA \cdot PB = PC \cdot PD$
- **Tangent and Secant:** $PT^2 = PA \cdot PB$
- **Intersecting chords**: If chords AE and BD intersect at E, then: $AE \cdot EC = BE \cdot ED$



9.6 Inscribed Right Triangle

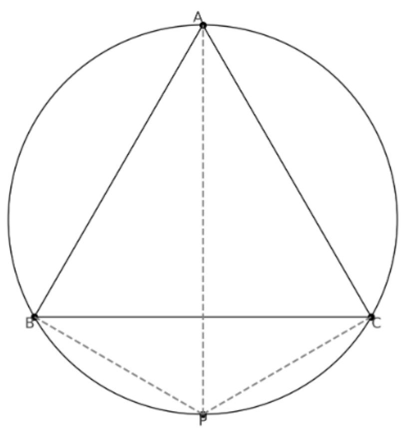
- If a triangle is inscribed in a circle and one side is a diameter, then the triangle is a right triangle.
- **Thales' Theorem:** $\angle ABC = 90°$ if AC is a diameter.



9.7 Van Schooten's Theorem

Theorem 9.1. *For point P on the minor arc $\overset{\frown}{BC}$ of equilateral triangle $\triangle ABC$, the sum of distances from P to B and C equals the distance from P to A:*

$$PA = PB + PC$$



9.8 Additional Tips

- Arcs are measured in degrees (or radians), but formulas often use radius-based calculations.
- Drawing auxiliary lines (like radii or diameters) often helps solve circle problems.
- Look for isosceles triangles formed by radii to simplify problems.

10 Constructions and Loci (February/March)

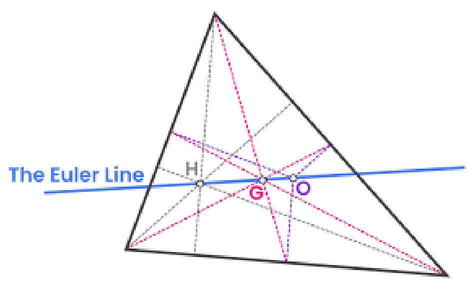
10.1 Triangle Centers

Center	Definition
Centroid	Intersection of the three medians (vertex to midpoint of opposite side)
Orthocenter	Intersection of the three altitudes (vertex perpendicular to opposite side)
Circumcenter	Intersection of the three perpendicular bisectors of the sides
Incenter	Intersection of the three angle bisectors

10.2 Euler Line

In any non-equilateral triangle, the **centroid**, **orthocenter**, and **circumcenter** lie on a straight line called the **Euler Line**.

$$\text{Let } G = \text{centroid}, \quad H = \text{orthocenter}, \quad O = \text{circumcenter}$$
$$\text{Then } HG = 2GO$$

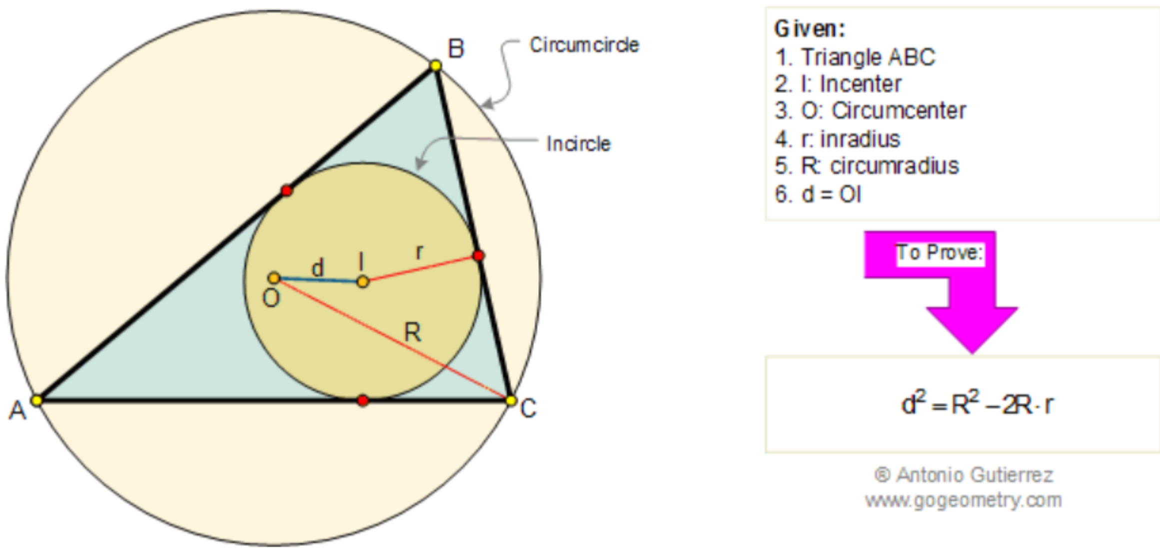


10.3 Euler's Geometry Theorem

In any triangle with circumradius R and inradius r, the distance d between the circumcenter and incenter satisfies:

$$d^2 = R(R - 2r)$$

This relationship provides a geometric link between the circumcenter and incenter.

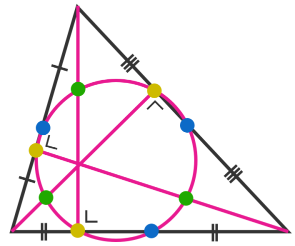


10.4 Nine-Point Circle

The nine-point circle of a triangle passes through:

- The midpoints of the three sides
- The feet of the three altitudes
- The midpoints of the segments from each vertex to the orthocenter

This circle's center lies on the Euler line, halfway between the orthocenter and circumcenter.



10.5 Locus Definitions

The **locus** is a set of points that satisfy a given condition. Common geometric loci include:

- **Circle:** Set of all points at a fixed distance from a given point (center).

- **Perpendicular Bisector:** Set of points equidistant from two given points.
- **Angle Bisector:** Set of points equidistant from two intersecting lines.
- **Parabola:** Set of points equidistant from a fixed point (focus) and a fixed line (directrix).

11 Areas of Plane Figures (February/March)

11.1 Area Formulas

Shape	**Area Formula**
Equilateral Triangle	$\dfrac{s^2\sqrt{3}}{4}$
Triangle (general)	$\dfrac{1}{2}bh$ or $\sqrt{s(s-a)(s-b)(s-c)}$
Parallelogram	bh
Rectangle	lw
Square	s^2
Rhombus / Kite	$\dfrac{1}{2}d_1d_2$
Trapezoid	$\dfrac{(b_1+b_2)h}{2}$
Regular Pentagon	$\dfrac{1}{4}\sqrt{25+10\sqrt{5}}s^2$
Regular Hexagon	$\dfrac{3s^2\sqrt{3}}{2}$
Regular Octagon	$2s^2(1+\sqrt{2})$
Cyclic Quadrilateral	$\sqrt{(s-a)(s-b)(s-c)(s-d)}$

11.2 Circle Area and Related Formulas

- **Circle**: $A = \pi r^2$
- **Sector Area (central angle θ in degrees)**:

$$A = \frac{\theta}{360^\circ} \cdot \pi r^2$$

- **Arc Length**:

$$L = \frac{\theta}{360^\circ} \cdot 2\pi r$$

11.3 Inradius and Circumradius

- **Inradius** (r) is the radius of the circle inscribed in a triangle—touching all three sides.

$$r = \frac{A}{s} \quad \text{where } A = \text{area},\ s = \text{semiperimeter}$$

- **Circumradius** (R) is the radius of the circle that passes through all three vertices.

$$R = \frac{abc}{4A} \quad \text{where } a, b, c = \text{sides of triangle, and } A = \text{area}$$

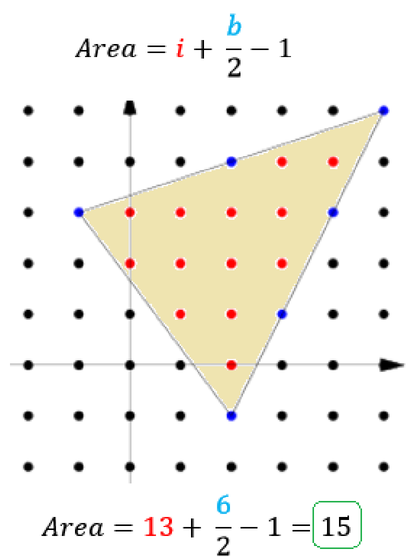
11.4 Pick's Theorem

For lattice polygons (whose vertices lie on grid points):

$$A = I + \frac{B}{2} - 1$$

where:

- I = number of interior lattice points
- B = number of boundary lattice points
- A = total area



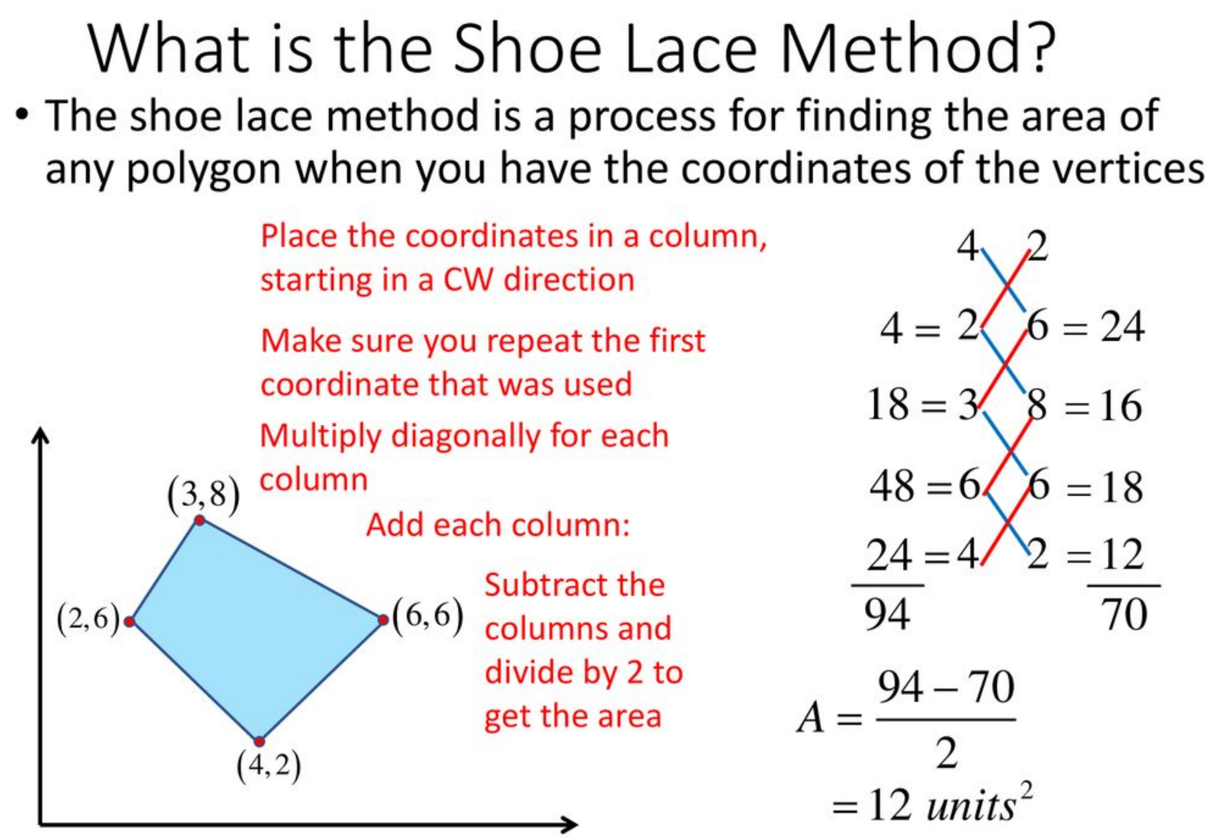
11.5 Shoelace Formula

For polygons with vertices $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the area is:

$$A = \frac{1}{2}\left|\sum_{i=1}^{n}(x_i y_{i+1} - x_{i+1} y_i)\right|$$

with the convention that $(x_{n+1}, y_{n+1}) = (x_1, y_1)$

Use when: You have coordinates of all vertices listed in order (clockwise or counterclockwise).



12 Lateral Area, Surface Area, and Volume (March)

12.1 Definitions

Before using the formulas, know these variables:

- B: Area of the base
- P: Perimeter of the base
- h: Height of the solid (distance between bases)
- l: Slant height (used for cones and pyramids)
- r: Radius of the base
- R: Radius of the larger base (used in frustums)

12.2 Formulas for Common Solids

Solid	**Lateral Surface Area (LSA)**	**Total Surface Area (TSA)**	**Volume**
Prism	Ph	$Ph + 2B$	Bh
Cylinder	$2\pi rh$	$2\pi r(h + r)$	$\pi r^2 h$
Pyramid	$\frac{1}{2}Pl$	$\frac{1}{2}Pl + B$	$\frac{1}{3}Bh$
Cone	πrl	$\pi r(l + r)$	$\frac{1}{3}\pi r^2 h$
Sphere	–	$4\pi r^2$	$\frac{4}{3}\pi r^3$
Frustum (Cone)	–	–	$\frac{\pi h}{3}(R^2 + Rr + r^2)$
Frustum (Pyramid)	–	–	$\frac{h}{3}(A + B + \sqrt{AB})$

12.3 Volume of Any Frustum

The volume V of any frustum with height H and base areas S_1 and S_2 is given by:

$$V = \frac{H}{3}\left(S_1 + S_2 + \sqrt{S_1 S_2}\right)$$

where:

- H = height of the frustum
- S_1 = area of the lower base
- S_2 = area of the upper base

12.4 Platonic Solids

A **Platonic Solid** is a regular, convex polyhedron where:

- All faces are congruent regular polygons
- The same number of faces meet at each vertex

There are only 5 Platonic Solids:

Name	Faces	Vertices	Edges	Face Type	Angle Deficit	Surface Area	Volume
Tetrahedron	4	4	6	Equilateral Triangle	720°	$\sqrt{3}s^2$	$\frac{s^3}{6\sqrt{2}}$
Cube	6	8	12	Square	360°	$6s^2$	s^3
Octahedron	8	6	12	Equilateral Triangle	240°	$2\sqrt{3}s^2$	$\frac{\sqrt{2}}{3}s^3$
Dodecahedron	12	20	30	Regular Pentagon	0°	$3\sqrt{25+10\sqrt{5}}\,s^2$	$\frac{1}{4}(15+7\sqrt{5})s^3$
Icosahedron	20	12	30	Equilateral Triangle	120°	$5\sqrt{3}s^2$	$\frac{5}{12}(3+\sqrt{5})s^3$

12.5 Euler's Formula

Euler's Theorem applies to convex polyhedra (solids with flat faces):

$$V - E + F = 2$$

where:

- V = number of vertices
- E = number of edges
- F = number of faces

12.6 Similar Solids

For solids that are similar (same shape, different size), with scale factor k:

- **Length ratio**: k
- **Surface area ratio**: k^2
- **Volume ratio**: k^3

12.7 Space Diagonals

- Rectangular Prism: $\sqrt{l^2 + w^2 + h^2}$
- Cube: $s\sqrt{3}$

13 Coordinate Geometry

13.1 Distance Formulas

- **Between two points (2D):**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Between two points (3D):**

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Point to line (in 2D):** For line $ax + by + c = 0$ and point (x_0, y_0),

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

- **Point to plane (in 3D):** For plane $ax + by + cz + d = 0$ and point (x_0, y_0, z_0),

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

13.2 Midpoint Formula

Midpoint M of segment AB where $A = (x_1, y_1)$ and $B = (x_2, y_2)$:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

13.3 Weighted Average (Weighted Point) Formula

Given points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ with weights w_1 and w_2, the weighted point P dividing the segment in ratio $w_2 : w_1$ is:

$$P = \left(\frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}, \quad \frac{w_1 y_1 + w_2 y_2}{w_1 + w_2}\right)$$

13.4 Slope Formula

Slope of line through points (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

13.5 Equation of a Circle

Circle with center (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

13.6 Centroid Formula

Centroid G of triangle with vertices $A = (x_1, y_1), B = (x_2, y_2), C = (x_3, y_3)$:

$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

13.7 Two Poles Problem

Two poles of heights a and b stand vertically apart. Lines are drawn from the top of each pole to the bottom of the other. The height h of the intersection of these lines above the ground is given by:

$$h = \frac{ab}{a + b}$$

Explanation: The intersection divides the segment in the harmonic mean of the two heights.