Important Instructions for this Test: The answer choice E: NOTA indicates that "none of these answers are correct." At some point the approximation $\pi \approx 3.14$ may be of interest. Problems are not necessarily in order of increasing difficulty, so don't be afraid to skip around. Good luck, and have fun!

1. Points L, E, G, O, S, I lie on a line in that order such that $LG = 4$, $EO = 5$, $GS = 5$, $OI = 6$, $LI = 14$. Find ES.

A: 4 B: 6 C: 8 D: 9 E: NOTA

2. Bill rolls a fair 20-sided die with faces labeled 1-20. Find the probability he rolls a number between 7 and 11 inclusive.

A: $\frac{1}{10}$ B: $\frac{1}{5}$ C: $\frac{1}{4}$ D: $\frac{3}{20}$ E: NOTA

3. Haru has some nonzero number of potted plants. She notices that she can split them evenly into groups of 4, groups of 5, or groups of 6. What is the smallest possible number of potted plants Haru could have?

A: 60 B: 15 C: 30 D: 120 E: NOTA

4. Let $f(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots + 2022x^{2021}$. Compute $f(-1)$.

A: 1011 B: 2022 C: -1011 D: -2022 E: NOTA

5. Collot starts with a positive integer, multiplies it by 3, adds 7, and then divides it by 2, resulting in the number 17. What positive integer did Collot start with?

A: 14 B: 29 C: 9 D: 5 E: NOTA

6. A certain cube satisfies the property that if its side length were doubled, its volume would increase by 21. What is the side length of this cube?

A: 3 B: $\sqrt[3]{3}$ C: $\sqrt{3}$ D: $\sqrt[3]{21}$ E: NOTA

7. Jack repeatedly flips a fair coin until he gets his first heads. What is the probability it takes him at least three flips to do so?

A: $\frac{1}{4}$ B: $\frac{1}{8}$ C: $\frac{1}{6}$ D: $\frac{1}{3}$ E: NOTA

8. Find the largest value of x such that $(x, 0)$ is equidistant from $(2,1)$ and the line $y = -2$.

A: $2 + \sqrt{3}$ B: 3 C: $2 + \sqrt{5}$ D: $2 + \frac{\sqrt{6}}{2}$ E: NOTA

9. A point is chosen uniformly at random on the interior of a sphere of radius 2022. What is the probability the point is closer to the surface of the sphere than it is to the center of the sphere?

A: $\frac{3}{4}$ B: $\frac{1}{4}$ C: $\frac{7}{8}$ D: $\frac{1}{8}$ E: NOTA

10. If $\sin(\alpha) = -\frac{5}{13}$ and $\cos(\beta) = \frac{4}{5}$, find the smallest possible positive value of $\cos(\alpha + \beta)$.

A: $\frac{63}{65}$ B: $\frac{16}{65}$ C: $\frac{33}{65}$ D: $\frac{52}{65}$ E: NOTA

11. Triangle ABC has $m\angle A = 36°$, $m\angle B = 48°$, $m\angle C = 96°$. Point D lies on $\overline{BC}$ such that $m\angle BAD = m\angle CAD$. Compute $|m\angle BDA - m\angle CDA|$.

A: 48° B: 12° C: 60° D: 0° E: NOTA

12. An ellipse satisfies the property that the endpoints of its minor axis and its foci all lie on the same circle. What is the eccentricity (the ratio of the distance between the foci of the ellipse to the length of the major axis) of such an ellipse?

A: $\frac{1}{2}$ B: $\frac{\sqrt{2}}{2}$ C: $\frac{\sqrt{3}}{3}$ D: 1 E: NOTA

13. Let $ABCD$ be a square of side length 3. Points P, Q, R, S lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, such that $AP = BQ = CR = DS = 1$. Find the ratio of the area of square $PQRS$ to the area of square $ABCD$.

A: $\frac{1}{3}$ B: $\frac{4}{9}$ C: $\frac{5}{9}$ D: $\frac{2}{3}$ E: NOTA

14. The polynomial $x^4 + 1$ can be factored as $(x^2 + ax + b)(x^2 + cx + d)$, where a, b, c, d are real numbers. Compute $a^2 + b^2 + c^2 + d^2$.

A: 4 B: 8 C: 10 D: 6 E: NOTA

15. Find the smallest integer value of n such that $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{n-1}+\sqrt{n}} > 10$.

A: 121 B: 101 C: 100 D: 122 E: NOTA

16. Pentagon $ABCDE$ has $m\angle A = m\angle B = m\angle D = 90°$ and $m\angle C = m\angle E = 135°$. If $EA = 4$, $AB = 6$, and $BC = 8$, find $CD + DE$.

A: $8\sqrt{2}$ B: $6\sqrt{2}$ C: $2\sqrt{13}$ D: $4\sqrt{2}$ E: NOTA

17. Find the maximum possible value of $4\cos\left(x+\frac{\pi}{3}\right)+6\cos\left(x-\frac{\pi}{3}\right)$ for real x.

A: 10 B: $2\sqrt{13}$ C: $2\sqrt{7}$ D: $2\sqrt{19}$ E: NOTA

18. Let f be the monic (leading coefficient 1) polynomial of minimal degree with rational coefficients that has $1+\sqrt[3]{3}$ as a root. Find the sum of the coefficients of f.

A: -3 B: 5 C: 11 D: 3 E: NOTA

19. The number 4^9+9^4 has three distinct prime factors. Find the sum of the digits of the largest of these three prime factors.

A: 17 B: 7 C: 11 D: 13 E: NOTA

20. Let M and N be points on sides $\overline{AB}$ and $\overline{AC}$ of triangle ABC respectively such that $\overline{MN} \parallel \overline{BC}$. If $AM=MN=NC=4$ and $AN=5$, find BC.

A: $\frac{32}{5}$ B: 8 C: $\frac{36}{5}$ D: 9 E: NOTA

21. Find the number of surjective non-decreasing functions from $\{1,2,3,4,5,6,7,8,9\}$ to $\{1,2,3\}$. (A function $f: X \to Y$ is *surjective* if for all y in Y there is some x in X with $f(x)=y$. A function $f: X \to Y$ is *non-decreasing* if for all x in X and y in X, $x \le y \Rightarrow f(x) \le f(y)$.)

A: 36 B: 28 C: 56 D: 84 E: NOTA

22. Let $P(x)$ be a polynomial with real coefficients such that $P\big(P(x)\big)=64x^4-96x^3+184x^2-111x+90$ for all x. Find $P(2)$.

A: 27 B: 15 C: 17 D: 5 E: NOTA

23. There exist four matrices A satisfying $A^2=\begin{pmatrix}4 & 10\\ 0 & 9\end{pmatrix}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 9\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^{-1}$. Find the maximum possible sum of the squares of the entries of such an A.

A: 92 B: 113 C: 17 D: 78 E: NOTA

24. Find the number of rational numbers r such $12r-\frac{18}{r}$ is an integer.

A: 36 B: 16 C: 72 D: 32 E: NOTA

25. Cyclic hexagon $ABCDEF$ has $AB = BC = CD = 3$ and $DE = EF = FA = 6$. Find the area of its circumscribed circle.

A: 27π B: 18π C: 21π D: 24π E: NOTA

26. The nth *harmonic number*, denoted H_n, is given by $H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Which of the following is equivalent to $H_1 + H_2 + H_3 + \cdots + H_{20}$?

A: $21H_{21} + 1$
B: $21H_{21} + 21$
C: $21H_{21} - 1$
D: $21H_{21} - 21$
E: NOTA

27. Find the greatest integer less than or equal to $\left(1 + \sqrt{3}\right)^8$.

A: 3104 B: 3105 C: 3103 D: 3102 E: NOTA

28. If $\sum_{n=0}^{\infty} \frac{\cos(\pi n/6)}{2^n}$ can be expressed as $a + b\sqrt{3}$ for rational a and b, find $a + b$.

A: $\frac{4}{13}$ B: $\frac{17}{13}$ C: $\frac{11}{13}$ D: $\frac{8}{13}$ E: NOTA

29. Let L be the set of all line segments in the xy-plane of length 8 with one endpoint on the x-axis and one endpoint on the y-axis. The set of midpoints of segments in L forms a closed curve. Find the greatest integer less than or equal to the area enclosed by this curve.

A: 13 B: 50 C: 12 D: 32 E: NOTA

30. Louis has a magical fair 20-sided die with faces labeled 1-20 with the following property: if Louis rolls the die and gets a number $n \geq 2$, the die disappears and is replaced with a fair n-sided die with faces labeled 1-n (with these same magical properties). If he rolls a 1, the die disappears, never to be seen again. If Louis starts with the 20-sided die and rolls the resulting sequence of dice until they disappear for good, what is the expected number of times he will roll the dice? (See problem 26 for H_n notation.)

A: $H_{21} + 1$ B: $H_{22} + 1$ C: $H_{19} + 1$ D: $H_{20} + 1$ E: NOTA