Answer Key:

1. B
2. C
3. A
4. C
5. C

6. B
7. A
8. A
9. C
10. C

11. A
12. B
13. C
14. D
15. D

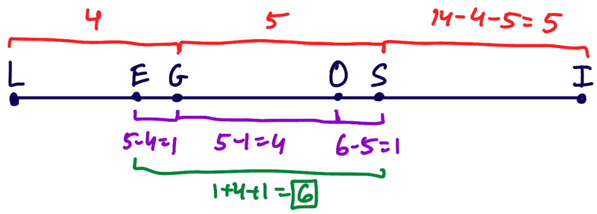
16. B
17. C
18. A
19. A
20. C

21. B
22. B
23. B
24. D
25. C

26. D
27. C
28. B
29. B
30. C

Solutions:

1. B: First note that $SI = LI - LG - GS = 14 - 4 - 5 = 5$, and then work your way left as shown below.



2. C: Bill could roll 7,8,9,10,11, for 5 successful options out of 20 total options, yielding a probability of $\boxed{\frac{1}{4}}$

3. A: This is simply the least common multiple of 4,5,6, which is $2^2 \cdot 5 \cdot 3 = \boxed{60}$.

4. C: Plugging in -1 gives $1 - 2 + 3 - 4 + \cdots - 2022 = (1 - 2) + (3 - 4) + \cdots + (2021 - 2022)$, which is $\frac{2022}{2} = 1011$ copies of -1, for a sum of $\boxed{-1011}$.

5. C: Prior to dividing by 2, he had $17 \cdot 2 = 34$. Prior to adding 7, he had $34 - 7 = 27$. Finally, prior to multiplying by 3, he had $\frac{27}{3} = \boxed{9}$.

6. B: Let s be the side length of the cube. The condition becomes $(2s)^3 = s^3 + 21$, or $7s^3 = 21$. This has solution $s = \boxed{\sqrt[3]{3}}$.

7. A: He needs the first two flips to end up tails, but after that we don't really care what happens (well, we want him to get heads *eventually*, but the probability that doesn't happen is 0). The probability of the first two flips landing heads is simply $\frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$.

8. A: The distance from $(x, 0)$ to $(2,1)$ is $\sqrt{(x-2)^2 + 1^2}$, and the distance from $(x, 0)$ to $y = -2$ is 2. We thus want to solve $\sqrt{(x-2)^2 + 1} = 2 \Rightarrow x^2 - 4x + 1 = 0$. There are many ways to solve this equation, but here's a pretty cool/instructive one. By Vieta's the sum of the roots is 4, so the roots take the form $2 + c$, $2 - c$ for some $c \geq 0$. The product of the roots is $1 = (2 + c)(2 - c) \Rightarrow 4 - c^2 = 1 \Rightarrow c = \sqrt{3}$. The largest value of x is then $\boxed{2 + \sqrt{3}}$. (https://www.poshenloh.com/quadraticdetail/ for more info on this method)

9. C: Suppose the point lies a distance d from the center of the circle. The closest point on the surface of the sphere is the endpoint of the radius passing through our selected point, so the distance to the surface of the sphere is $2022 - d$. We thus need $2022 - d \geq d \Rightarrow d \geq 1011$. This means we can pick any point inside the original sphere but outside the sphere of radius 1011 concentric with the original sphere. This yields a probability of $\frac{\frac{4}{3}\pi \cdot 2022^3 - \frac{4}{3}\pi \cdot 1011^3}{\frac{4}{3}\pi \cdot 2022^3} = \frac{8-1}{8} = \boxed{\frac{7}{8}}$.

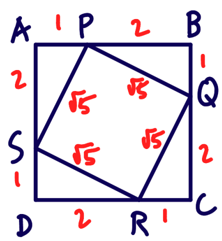
10. C: Using the identity $\sin^2(\theta) + \cos^2(\theta) = 1$, we can compute $\cos(\alpha) = \pm\frac{12}{13}$ and $\sin(\beta) = \pm\frac{3}{5}$. We can then compute $\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) = \left(\pm\frac{12}{13}\right)\left(\frac{4}{5}\right) - \left(-\frac{5}{13}\right)\left(\pm\frac{3}{5}\right)$. To make this positive but as small as possible, we need the first term to be positive but the second term to be negative. Choosing the signs appropriately yields $\frac{48}{65} - \frac{15}{65} = \boxed{\frac{33}{65}}$.

11. A: We compute $m\angle BDA = 180° - m\angle DBA - m\angle BAD$ and $m\angle CDA = 180° - m\angle DCA - m\angle CAD$. Recalling that $m\angle BAD = m\angle CAD$, we have

$$m\angle BDA - m\angle CDA = (180° - m\angle DBA - m\angle BAD) - (180° - m\angle DCA - m\angle CAD)$$
$$= -48° - (-96°) = \boxed{48°} \qquad \text{(red terms cancel)}$$

12. B: Letting $2a$ be the length of the major axis, $2b$ be the length of the minor axis, and $2c$ be the distance between the foci, the given condition is equivalent to $b = c$. Using the relationship $c^2 = a^2 - b^2$, we can compute $a = \sqrt{2}c$. The eccentricity is then $\frac{2c}{2a} = \frac{2c}{2\sqrt{2}c} = \boxed{\frac{\sqrt{2}}{2}}$.

13. C: We can easily compute $PB = QC = RD = SA = 3 - 1 = 2$. The Pythagorean theorem then yields $PQ = QR = RS = SQ = \sqrt{1^2 + 2^2} = \sqrt{5}$. The area of $PQRS$ is thus $\left(\sqrt{5}\right)^2 = 5$, while the area of $ABCD$ is $3^2 = 9$, so the desired ratio is $\boxed{\frac{5}{9}}$ (in fact, this ratio is independent of the side length of the square).



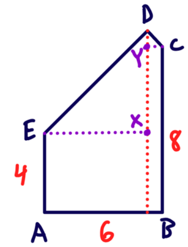
14. D: This problem can be solved by direct expansion, matching coefficients, and solving the resulting system of equations, but here's a more instructive solution. The roots of $x^4 + 1$ are the primitive eighth roots of unity, namely $e^{k\pi i/4}$ for $k = 1,3,5,7$. Each of the quadratics in the factorization will have two roots, which must be complex conjugates for the polynomials to have real coefficients. In particular, one term will have roots $e^{\pi i/4}$ and $e^{7\pi i/4}$, while the other will have roots $e^{3\pi i/4}$ and $e^{5\pi i/4}$. By Vieta's (backwards), the former will be $x^2 - \sqrt{2}x + 1$ and the latter will be $x^2 + \sqrt{2}x + 1$, yielding an answer of $\left(-\sqrt{2}\right)^2 + 1^2 + \left(\sqrt{2}\right)^2 + 1^2 = \boxed{6}$.

15. D: Consider an arbitrary term $\frac{1}{\sqrt{k}+\sqrt{k+1}}$. Multiplying numerator and denominator by $\sqrt{k+1} - \sqrt{k}$ yields $\frac{1}{\sqrt{k+1}+\sqrt{k}} \cdot \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}} = \frac{\sqrt{k+1}-\sqrt{k}}{(k+1)-k} = \sqrt{k+1} - \sqrt{k}$. Therefore, the sum in question becomes

$$\left(\sqrt{2} - \sqrt{1}\right) + \left(\sqrt{3} - \sqrt{2}\right) + \cdots + \left(\sqrt{n} - \sqrt{n-1}\right) = \sqrt{n} - \sqrt{1}$$

We thus want $\sqrt{n} > 11$, which first occurs when $n = \boxed{122}$.

16. B: Considering the diagram below, drop a perpendicular from D to $\overline{AB}$, and then drop perpendiculars from E and C to this perpendicular. Noting the 45-45-90 triangles formed, we can see that $DE = \sqrt{2}EX$ and $DC = \sqrt{2}CY$. We can then compute $CD + DE = \sqrt{2}(CY + EX) = \sqrt{2}(AB) = \boxed{6\sqrt{2}}$.

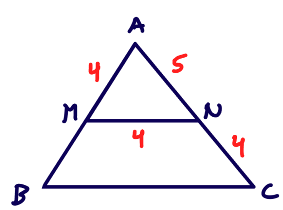


17. C: We first compute $4\cos\left(x + \frac{\pi}{3}\right) = 4\cos(x)\cos\left(\frac{\pi}{3}\right) - 4\sin(x)\sin\left(\frac{\pi}{3}\right)$ and $6\cos\left(x - \frac{\pi}{3}\right) = 6\cos(x)\cos\left(-\frac{\pi}{3}\right) - 4\sin(x)\sin\left(-\frac{\pi}{3}\right)$. Adding and simplifying, this becomes $5\cos(x) + \sqrt{3}\sin(x)$. Noting that $\sqrt{5^2 + (\sqrt{3})^2} = 2\sqrt{7}$, we can rewrite this as $2\sqrt{7}\left(\frac{5}{2\sqrt{7}}\cos(x) + \frac{\sqrt{3}}{2\sqrt{7}}\sin(x)\right) = 2\sqrt{7}\sin\left(\sin^{-1}\left(\frac{5}{2\sqrt{7}}\right) + x\right)$, which clearly has maximum value $\boxed{2\sqrt{7}}$.

18. A: We have $x = 1 + \sqrt[3]{3} \Rightarrow x - 1 = \sqrt[3]{3} \Rightarrow (x-1)^3 = 3 \Rightarrow x^3 - 3x^2 + 3x - 4 = 0$. We claim this is the desired polynomial of minimal degree. If not, then it would need to factor into a linear term and a quadratic term. For this to happen, the linear term would be of the form $x - r$, where r is a root of the polynomial. Inspecting the graph of $(x-1)^3 - 3$, we see that $1 + \sqrt[3]{3}$ is its only root, which is not rational, so no such factorization can happen. This yields an answer of $1 - 3 + 3 - 4 = \boxed{-3}$.

19. A: From Sophie-Germain, $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$. Noting that $9^4 + 4^9 = 9^4 + 4(16)^4$, we can then factor as $\left(9^2 + 2(9)(16) + 2(16^2)\right)\left(9^2 - 2(9)(16) + 2(16^2)\right) = (881)(305) = 5 \cdot 61 \cdot 881$. This yields a final answer of $8 + 8 + 1 = \boxed{17}$.

20. C: The parallel lines give $\angle AMN \cong \angle ABC$ and $\angle ANM \cong \angle ACB$, so $\Delta AMN \sim \Delta ABC$ by AA. We can then say $\frac{BC}{MN} = \frac{AC}{AN} \Rightarrow BC = \frac{9 \cdot 4}{5} = \boxed{\frac{36}{5}}$.



21. B: Note that we must have $f(1) = 1$ (else nothing can equal 1 and it's not surjective). Note further that an increasing surjection can be uniquely determined by the points where it increases (e.g., if $1,2,3 \mapsto 1$, $4,5,6 \mapsto 2$, and $7,8,9 \mapsto 3$, we can encode this function uniquely by the pair (4,7)). It thus suffices to pick two points where the function increases. We can pick any two of $\{2,3,\ldots,9\}$, for a total of $\binom{8}{2} = \boxed{28}$ options.

22. B: Note that if the leading term of P has degree r, then the leading term of $P \circ P$ has degree r^2 (since we raise this leading term to the same power). Thus P is a quadratic polynomial, so we can let $P(x) = ax^2 + bx + c$. We can then compute $P(P(x)) = a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c$. Looking at the x^4 term, we get $a^3 = 64 \Rightarrow a = 4$. Looking at the x^3 term, we get $2a^2b = -96 \Rightarrow b = -3$. Finally, looking at the x term, we get $2abc + b^2 = -111 \Rightarrow c = 5$. To check, we can expand

$$4(4x^2 - 3x + 5)^2 - 3(4x^2 - 3x + 5) + 5 = 64x^4 - 96x^3 + 184x^2 - 111x + 90,$$

As desired. Thus $P(2) = 4(2)^2 - 3(2) + 5 = \boxed{15}$.

23. B: The latter factorization of A^2 is of the form PDP^{-1} for appropriate P and D. If you squint hard enough, it's actually of the form PD^2P^{-1} for $D = \begin{pmatrix} \pm 2 & 0 \\ 0 & \pm 3 \end{pmatrix}$ and the same P as before. Noting that $PD^2P^{-1} = PDP^{-1}PDP^{-1} = (PDP^{-1})^2$, this gives us our four square roots, namely PDP^{-1} for the four options of D as shown above. Noting that negatives will yield the same sum of squares, we only need to check two:

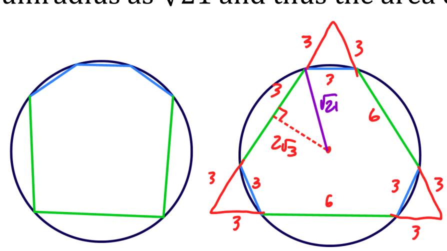
$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 6 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 3 \end{pmatrix} \Rightarrow 2^2 + 2^2 + 0^2 + 3^2 = 17$$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -6 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -10 \\ 0 & -3 \end{pmatrix} \Rightarrow 2^2 + (-10)^2 + (-3)^2 = 113$$

Of these, the larger is $\boxed{113}$.

24. D: We want to find rational r such that $12r - \frac{18}{r} = m$ for some integer m, or equivalently $12r^2 - mr - 18 = 0$. This is a polynomial with integer coefficients, so by the rational root theorem the only possible rational roots take the form $\pm\frac{a}{b}$ where a is a factor of 18 and b is a factor of 12. To count the possible values of r, we consider prime factors. The power of 2 could be $\{1,0,-1,-2\}$, and the power of 3 could be $\{2,1,0,-1\}$. Also considering the $\pm$, this yields $4 \cdot 4 \cdot 2 = 32$ candidates for r. We claim each such r yields an integer. To see this, let $r = \pm\frac{a}{b}$, so $12r - \frac{18}{r} = \pm\frac{12a}{b} \mp \frac{18b}{a}$. Since b divides 12 and a divides 18, this is a sum of two integers and thus an integer. So, the answer is $\boxed{32}$.

25. C: The first observation to make is that you can flip the ordering of the sides of a cyclic polygon without affecting the circumcircle of said polygon (if it helps, imagine rearranging the arcs cut off by each side). Doing said flipping to get alternating sides of length 3 and 6, we can see by symmetry that the resulting hexagon has interior angles of measure 120°. We can then extend the long sides of this hexagon to get an equilateral triangle of side length 12 and thus inradius $2\sqrt{3}$. Dropping a perpendicular and using the Pythagorean theorem gives the circumradius as $\sqrt{21}$ and thus the area of the circumcircle as $\boxed{21\pi}$.



26. D: We first compute the first few harmonic numbers and sums:

- $H_1 = 1$
- $H_2 = 1 + \frac{1}{2} = \frac{3}{2}$
- $H_3 = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$
- $H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{50}{24}$
- $H_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{274}{120}$

- $H_1 = 1$
- $H_1 + H_2 = 1 + \frac{3}{2} = \frac{5}{2}$
- $H_1 + H_2 + H_3 = 1 + \frac{3}{2} + \frac{11}{6} = \frac{26}{6}$
- $H_1 + H_2 + H_3 + H_4 = 1 + \frac{3}{2} + \frac{11}{6} + \frac{50}{24} = \frac{154}{24}$
- $H_1 + H_2 + H_3 + H_4 + H_5 = 1 + \frac{3}{2} + \frac{11}{6} + \frac{50}{24} + \frac{274}{120} = \frac{994}{120}$

Staring at this for a bit (and perhaps taking inspiration from the answer choices) we note that $H_k - \frac{1}{k}(H_1 + \cdots + H_{k-1}) = 1$, at least for small k. To see that this holds in general, we proceed by induction. The base case $k = 2$ is easy. Supposing it holds for k, we can see that

$$H_{k+1} - \frac{1}{k+1}(H_1 + \cdots + H_k) = H_{k+1} - \frac{1}{k+1}(\color{red}{kH_k - k} + H_k) = H_{k+1} - H_k + \frac{k}{k+1}$$
$$= H_k + \frac{1}{k+1} - H_k + \frac{k}{k+1} = 1,$$

completing the induction (where the red is from the induction hypothesis).

Thus, we know that $H_1 + \cdots + H_k = (k+1)(H_{k+1} - 1)$, for all k. Plugging in $k = 20$ gives $\boxed{21H_{21} - 21}$.

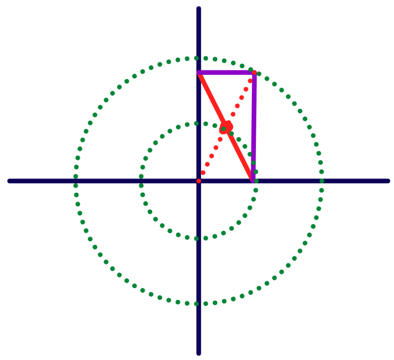
27. C: Let $a = 1 + \sqrt{3}$ and $b = 1 - \sqrt{3}$. We want to compute $\lfloor a^8 \rfloor$. We can easily compute $a + b = 2$ and $ab = -2$. We can then compute

$$a^2 + b^2 = (a+b)^2 - 2ab = 2^2 - 2(-2) = 8$$
$$a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = 8^2 - 2(-2)^2 = 56$$
$$a^8 + b^8 = (a^4 + b^4)^2 - 2a^4b^4 = 56^2 - 2(-2)^4 = 3104$$

Noting that $-1 < b < 0$, we see $0 < b^8 < 1$, so $\lfloor a^8 \rfloor = \boxed{3103}$.

28. B: Recall that $\cos(x) = \mathrm{Re}(e^{ix})$. We can thus rewrite the sum in question as $\sum_{n=0}^{\infty} \frac{\mathrm{Re}(e^{n\pi i/6})}{2^n} = \mathrm{Re} \sum_{n=0}^{\infty} \left(\frac{e^{\pi i/6}}{2}\right)^n$. This sum is simply a geometric series with sum $\frac{1}{1 - e^{\pi i/6}/2} = \frac{4}{4 - \sqrt{3} - i}$. Multiplying numerator and denominator by $4 - \sqrt{3} + i$ yields $\frac{4(4-\sqrt{3}+i)}{(4-\sqrt{3})^2 + 1} = \frac{16 - 4\sqrt{3} + 4i}{20 - 8\sqrt{3}} = \frac{4 - \sqrt{3} + i}{5 - 2\sqrt{3}}$. Taking the real part gives $\frac{4-\sqrt{3}}{5-2\sqrt{3}}$, and multiplying numerator and denominator by $5 + 2\sqrt{3}$ gives $\frac{(4-\sqrt{3})(5+2\sqrt{3})}{5^2 - (2\sqrt{3})^2} = \frac{14 + 3\sqrt{3}}{13}$, yielding a sum of $\frac{14}{13} + \frac{3}{13} = \boxed{\frac{17}{13}}$.

29. B: Consider the rectangle with three vertices at the origin and the endpoints of the segment in question. The other diagonal of this rectangle also has length 8. The key observation to make is that if instead of the original segment we were given this other diagonal, we could reconstruct the rectangle and thus our original segment. Furthermore, since the diagonals of a rectangle bisect each other, both diagonals share a midpoint. In essence, this means that we can simply consider the set of midpoints segments of length 8 with an endpoint at the origin. This, however, is simply a circle of radius 4, corresponding to an area of 16π. Using the approximation $\pi \approx 3.14$ (from the instructions), this gives us $16 \cdot 3.14 = 50.24$, which yields an answer of $\boxed{50}$.



30. C: Let E_n denote the expected number of remaining rolls for an n-sided die with this property. Let's start simpler with a $n = 2$. Rolling this die will yield a 1 with probability $\frac{1}{2}$, after which there will be 0 remaining rolls, or a 2 with probability $\frac{1}{2}$, after which there will be E_2 remaining rolls. Including the roll itself, we get $E_2 = \frac{1}{2}(0) + \frac{1}{2}E_2 + 1$. Similarly, we can compute $E_3 = \frac{1}{3}(0) + \frac{1}{3}E_2 + \frac{1}{3}E_3 + 1$, and in general, $E_n = \frac{1}{n}(0) + \frac{1}{n}E_2 + \cdots + \frac{1}{n}E_n + 1$. Solving the first few equations, we can get $E_2 = 2, E_3 = \frac{5}{2}, E_4 = \frac{17}{6}$. Staring at this a bit (and perhaps taking inspiration from the answer choices), it looks like $E_k = H_{k-1} + 1$, at least for small k. To see this holds for all k, we proceed by induction. $k = 2,3,4$ are already handled. In general, we have

$$E_{k+1} = \frac{k+1}{k}\left(\frac{1}{k+1}(0) + \frac{1}{k+1}E_2 + \cdots + \frac{1}{k+1}E_k + 1\right)$$

by rearranging the recurrence from before. By the induction hypothesis, this is equivalent to

$$\frac{1}{k}(H_1 + 1 + H_2 + 1 + \cdots + H_{k-1} + 1) + \frac{k+1}{k} = \frac{1}{k}(H_1 + \cdots + H_{k-1}) + \frac{k-1}{k} + \frac{k+1}{k}$$

Using the identity established in problem 26, this is equivalent to

$$\frac{1}{k} \cdot k(H_k - 1) + 2 = H_k + 1,$$

completing the induction. The desired answer is then $E_{20} = \boxed{H_{19} + 1}$.