Fall Interschool Test 2024

Important Instructions for this Test:
Write your answers in exact, simplified form, unless instructed otherwise.
In problems involving combinatorial games or combinatorial game theory, assume that the following normal convention applies: the player who makes the last move wins.
Good luck and have fun!

1. What is the set of all digits, if any, that never appear in the decimal expansion of $\frac{1}{81}$?

2. In any standard 5 by 5 magic square, what is the sum of the numbers in any column?

3. Let *m* and *n* represent non-random integers such that $0 < m < n$. In a one-dimensional symmetric random walk with step size 1, what algebraic expression, in simplified form, represents the probability of reaching *n* before reaching 0, starting at *m*?

4. In abstract algebra, what word or name (besides "commutative" or a form of this word) describes a group with an operation that has the commutative property?

5. What is the last name of the mathematician whose inequality states that for any random variable *X* with a mean and a standard deviation, and positive number *k*, the probability that *X* is within *k* standard deviations from its mean is at least $1 - \frac{1}{k^2}$?

6. What is the last name of the mathematician that is used to describe a sequence $\{a_n\}$ such that for any $\varepsilon > 0$, there exists a natural number *N* such that $|a_n - a_m| < \varepsilon$ whenever both $m > N$ and $n > N$?

7. In spherical geometry, what is the sum of the interior angles in a spherical triangle that encloses $\frac{1}{5}$ of the surface area of the sphere on which it lies? Express your answer in degrees.

8. In Vedic math calculations, sometimes bars are placed above digits, but these bars do not indicate repeating digits. In Vedic math, $83{,}00\bar{5},6\overline{21}$ is equivalent to what normal base-ten number?

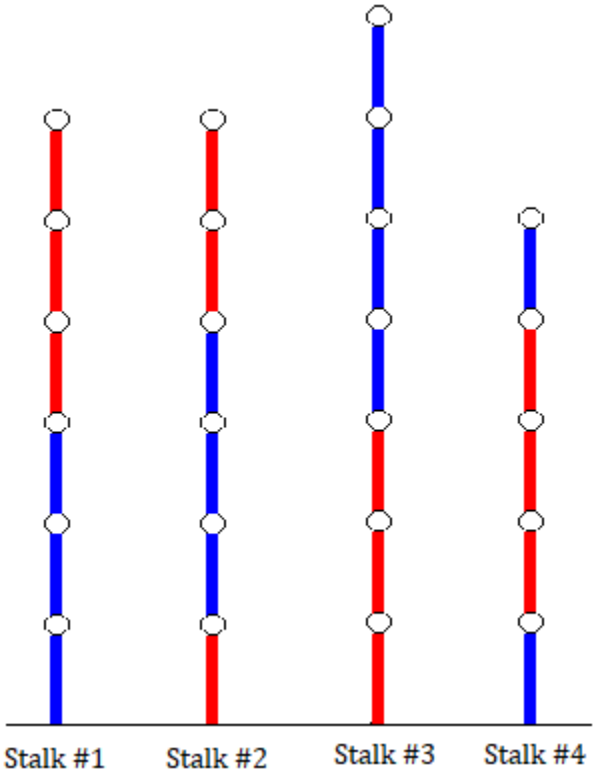
Each of problems 9 through 11, inclusive, describes a position in the combinatorial game of Nim. For each position, if there's a winning move for the next player, then specify how many coins that player should remove and the stack number from which that player should remove the coins. If there's no winning move for the next player, write "No winning move".

9. Stacks #1, #2, and #3 have 6, 7, and 8 coins, respectively.

10. Stacks #1, #2, and #3 have 5, 11, and 14 coins, respectively.

11. Stacks #1, #2, #3, and #4 have 9, 10, 11, and 12 coins, respectively.

Problems 12 and 13 pertain to the following configuration of red and blue segments from the combinatorial game Red-Blue Hackenbush.



12. What is the value of the position according to combinatorial game theory, counting blue as positive and red as negative?

13. If the correct answer to #12 is a positive number, then if Blue plays next, describe Blue's winning move.
If the correct answer to #12 is a negative number, then if Red plays next, describe Red's winning move.
If the correct answer to #12 is zero, then state whether the next player is winning or losing, regardless of which color plays next.

14. In the combinatorial game Red-Blue Hackenbush, consider a vertical stalk (chain) consisting of a blue segment on the bottom and 10 red segments on top. What is the value of n such that if n copies of this stalk and one copy of a stalk consisting of just one red segment are all placed side by side, then the player to go second (regardless of color) has a winning strategy?

The system of surreal numbers is an extension of the set of real numbers. In problems 15 through 20, inclusive, write the surreal number (which may or may not be real), if any, that is equivalent to the given ordered pair of lists. Numbers (if any) within each list are separated by commas, and the lists are separated by a vertical bar.

Write answers in simplified, exact form. If the ordered pair of lists is not equivalent to a surreal number, then write "None".

15. $\{-2 \mid 6\}$

16. $\{4 \mid 5\}$

17. $\left\{0 \middle| \frac{3}{8}\right\}$

18. $\{2, 4 \mid 3, 5\}$

19. $\{0, 1, 2, 3, 4, \ldots \mid \}$

20. $\{2, 5 \mid 10, 19\}$

21. List the eigenvalue(s), real and/or non-real, of the matrix $\begin{bmatrix} 5 & 3 \\ -7 & 15 \end{bmatrix}$.

22. Find the value of c for which the matrix $\begin{bmatrix} 10 & 20 \\ c & 30 \end{bmatrix}$ has exactly one distinct real eigenvalue.

23. If a 2 x 2 matrix A with real entries has trace 4 and determinant 5, then list the eigenvalue(s), real and/or non-real, of A.

24. Starting with a square including its interior, a fractal is constructed in infinitely many stages. In the first stage, the square is divided into an 8 by 8 grid of smaller congruent squares, then all except the four (smaller) corner squares are removed. In each successive stage, the same process is performed on each remaining smallest square. What is the dimension of this fractal?

25. Starting with a cube including its interior, a fractal is constructed in infinitely many stages. In the first stage, the cube is divided into an n by n by n grid of smaller congruent cubes, then all except the eight (smaller) corner cubes are removed. In each successive stage, the same process is performed on each remaining smallest cube, and n is the same positive integer in each stage. If the dimension of this fractal is 0.75, then what is the value of n?

In combinatorial game theory involving two players called Left and Right, the smallest type of advantage for Left is called a tiny, and the smallest type of advantage for Right is called a miny. Numbers are attached to tinies to distinguish them, and similarly for minies.
<u>In problems 26 through 29, inclusive,</u> for each sum of tinies and minies, write "L" if Left has a winning strategy regardless of who goes first, "R" if Right has a winning strategy regardless of who goes first, "F" if whoever goes first has a winning strategy, and "S" if whoever goes second has a winning strategy.

26. The sum of one copy of each of tiny 1 and miny 2

27. The sum of one copy of each of tiny 1, miny 2, miny 3, and tiny 4

28. The sum of ten copies of tiny 3 and one copy of miny 2

29. The sum of one copy of each of tiny 2, tiny 3, miny 2, and miny 3

30. In music, how many cents are there in an octave?

31. In music, if a pitch increases by one cent, then by what percent does its frequency increase? <u>Round your answer to the nearest ten-thousandth of a percent and include the percent symbol</u>.

32. What word describes a differential equation that can be written in the form $\frac{dy}{dx} = f(x)g(y)$?

33. What word describes a differential equation that can be written in the form $M(x,y)dx+N(x,y)dy=0$, where $M(x,y)$ and $N(x,y)$ are functions with continuous first partial derivatives such that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$?

34. A common theoretical model for pricing stock options, based on a differential equation, is most commonly named after the last names of two mathematicians. What is the name of this model?

35. A trick, named after the last name of a 20th century physicist, for finding $\int_0^\infty x^n e^{-ax}dx$ for positive *a* is based on observing that $\int_0^\infty x^n e^{-ax}dx=(-1)^n\int_0^\infty (-x)^n e^{-ax}dx=(-1)^n\int_0^\infty \frac{\partial^n}{\partial a^n}\left(e^{-ax}\right)dx=(-1)^n\frac{d^n}{da^n}\left(\int_0^\infty e^{-ax}dx\right)=(-1)^n\frac{d^n}{da^n}\left(\frac{1}{a}\right)=\frac{n!}{a^{n+1}}$. What is the name of this trick?

36. In complex analysis, a function $f(z)=u(x,y)+iv(x,y)$, where $z=x+iy$, is analytic at a point if and only if both $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$ at that point. This last pair of equations is named after the last names of two mathematicians. What is the name of this pair of equations?

Consider the following two-player combinatorial game: the position starts with one pile of counters, and players take turns removing exactly 2, 3, or 5 counters from the pile.
Each of problems 37 through 39, inclusive, describes a starting position. For each position, if there's a winning move for the next player, then write the number of counters that player should remove. If there's no winning move for the next player, write "No winning move".

37. A pile of 35 counters

38. A pile of 50 counters

39. A pile of 100 counters

40. Two volleyball teams, A and B, currently have a tied score in a game, with team A about to serve. The first team to go ahead by 2 points wins the game, and either the receiving or serving team can win any given point (rally scoring). Each team, when serving on a given point, wins that point with probability $\frac{1}{3}$ and otherwise allows the receiving team to win that point. The team that wins any given point then serves on the next point. The probability that a given team wins a given point depends <u>only</u> on whether or not that team is serving on that point. What is the probability that team A wins the game?

41. Solve the following variant of the previous problem: two volleyball teams, A and B, currently have a tied score in a game, with team A about to serve. The first team to go ahead by 2 points wins the game, and only the serving team can win points (side-out scoring). Each team, when serving on a given rally, wins that rally (and the point) with probability $\frac{1}{3}$ and otherwise allows the receiving team to win that rally (but not yet the point). The team that wins any given rally serves on the next rally. The probability that a given team wins a given rally depends <u>only</u> on whether or not that team is serving on that rally. What is the probability that team A wins the game?

42. A fair standard 6-sided die is rolled repeatedly until the sequence 1, 2 or the sequence 3, 3 occurs on two consecutive rolls. What is the probability that the sequence 1, 2 occurs on two consecutive rolls first?

43. A pair of fair standard 6-sided dice is rolled repeatedly. On each roll, the sum of the numbers on the pair of dice is recorded. The rolling continues until two consecutive sums of exactly 7 (rolling two 7's in a row) or two consecutive sums of exactly 8 (rolling two 8's in a row) occur. What is the probability that two consecutive sums of exactly 7 occur first?

44. Bag #1 contains *m* white balls, *n* black balls, and no other balls, where *m* and *n* are non-random positive integers and all balls are identical except for color. Andrew, who knows the number of balls of each color in bag #1, randomly draws balls one at a time without replacement from bag #1, until he draws all of the balls of one color. The probability that the last ball he draws is white is $\frac{3}{8}$. Bag #2 contains *m* white balls, 2*n* black balls, and no other balls; all balls are identical except for color. Beth, who knows the number of balls of each color in bag #2, randomly draws balls one at a time without replacement from bag #2, until she draws all of the balls of one color. What is the probability that the last ball she draws is white?

45. From a standard 52-card deck, how many possible 6-card hands are there, consisting of three pairs with a different value in each pair, if the order of the cards in the hand is unimportant?

46. In the standard Towers of Hanoi puzzle, let $f(n)$ represent the minimum number of moves needed for *n* discs, where *n* is a positive integer. What is the smallest possible positive integer value of *n* such that $f(n) \geq 1{,}000{,}000$?

47. If *A* represents the 2 x 2 matrix $\begin{bmatrix} 0.1 & 0.8 \\ 0.9 & 0.2 \end{bmatrix}$, then what 2 x 2 matrix is equal to $\lim_{n\to\infty} A^n$?

Problems 48 and 49 pertain to the combinatorial game called Contorted Fractions, involving two players called Left and Right. Consider the position consisting of the three fractions $\frac{1}{6}$, $\frac{1}{7}$, and $-\frac{3}{16}$.

48. What is the value of the position according to combinatorial game theory?

49. If the correct answer to #48 is a positive number, then if Left plays next, describe Left's winning move.
If the correct answer to #48 is a negative number, then if Right plays next, describe Right's winning move.
If the correct answer to #48 is zero, then state whether the next player is winning or losing, regardless of who plays next.

50. Bag #1 contains *m* white balls, *n* black balls, and no other balls, where *m* and *n* are non-random positive integers. Bag #2 contains just one white ball. All balls in both bags together are identical except for color. A ball is randomly drawn from bag #1 and transferred to bag #2, then a ball is randomly drawn from bag #2. Given that the ball drawn from bag #2 is white, what algebraic expression, in simplified form, represents the conditional probability that the remaining ball in bag #2 is white?

2025 State Convention Poster Topic and T-Shirt Design

POSTER TOPIC: The 2nd to last question is to give a suggestion(s) for the topic for the poster for the 2025 FAMAT State Convention. This question will not count as part of the test. Members of the FAMAT Board will review all suggestions and choose the topic. Schools may submit more than one poster topic suggestion. Please submit all poster topic ideas to the email below.

T-SHIRT/PROGRAM COVER: The last question is to design a cover/t-shirt design for the 2025 Florida Association of Mu Alpha Theta State Convention. Black and white, camera ready, pictures may be placed on standard white copy paper. A second copy of the same picture with suggested colors can also be turned in. This question will not count as part of the test, but will carry with it financial rewards for the winning schools. The school whose design is selected by the FAMAT Board for the front of the State Convention t-shirt will receive two free student registrations for the State Convention while the school whose design is selected for the back of the t-shirt will receive one free student registration. Schools may submit more than one design. Please send a PDF file or JPEG file for each design to the email below. Please save each PDF or JPEG file in the format described below.

The poster topic and t-shirt/program logos must be emailed by 11:59 pm November 6 to:

For the poster topics:
Derek Rampal
Email famatstateconvention@gmail.com
subject: 2025 State Convention Poster Topic Suggestions

For the t-shirt designs
Derek Rampal
Email: famatstateconvention@gmail.com
subject: 2025 State Convention T-Shirt Designs
Please remember to include contact information when sending your email.

For each t-shirt design attachment, please save it as:
SchoolName_ArtistName_BW_File# (change BW to Color for the color file).

Answer Sheet Page 1 – School Name:

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Answer Sheet Page 2 – School Name:

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